

Mickelsson algebras and Zhelobenko operators

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ABSTRACT

We construct a family of automorphisms of Mickelsson algebra, satisfying braid group relations. The construction uses 'Zhelobenko cocycle' and includes the dynamical Weyl group action as a particular case.

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1. INTRODUCTION

Mickelsson algebras were introduced in [M] for the study of Harish-Chandra modules of reductive groups. The Mickelsson algebra, related to a real reductive group, acts in the space of highest weight vectors of its maximal compact subgroup, and each irreducible Harish-Chandra module of the initial reductive group is uniquely characterized by this action.

A similar construction takes place for any associative algebra \mathcal{A} , which contains a universal enveloping algebra $U(\mathfrak{g})$ (or its q -analog) of a contragredient Lie algebra \mathfrak{g} with a fixed Gauss decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}$. Namely, we define the Mickelsson algebra $S^n(\mathcal{A})$ as the quotient of the normalizer $N(\mathcal{A}\mathfrak{n})$ of the ideal $\mathcal{A}\mathfrak{n}$ over the ideal $\mathcal{A}\mathfrak{n}$. In this case for any representation V of \mathcal{A} the Mickelsson algebra $S^n(\mathcal{A})$ acts in the space V^n of \mathfrak{n} -invariant vectors. This construction performs a reduction of a representation of \mathcal{A} over the action of $U(\mathfrak{g})$ and can be viewed as a counterpart of

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hamiltonian reduction. It was applied for various problems of representation theory, see the survey [T2] and references therein.

The structure of Mickelsson algebra simplifies after the localization over a certain multiplicative subset of $U(\mathfrak{h})$, where \mathfrak{h} is the Cartan subalgebra of \mathfrak{g} . The corresponding algebra $Z^n(\mathcal{A})$ is generated by a finite-dimensional space of generators, which obey quadratic-linear relations. These generators can be defined with the help of an extremal projector of Asherova-Smirnov-Tolstoy [AST]. An application of the extremal projector to the study of the Mickelsson algebras $Z^n(\mathcal{A})$ was developed by Zhelobenko [Zh]. Besides, Zhelobenko developed so called 'dual methods', where he gave a construction of another generators of the Mickelsson algebra by means of a family of special operators, which form a cocycle on the Weyl group [Zh2].

Later Mickelsson algebras regenerate in the theory of dynamical quantum groups. Their basic ingredients, intertwining operators between Verma modules and their tensor products with finite-dimensional representations actually form special Mickelsson algebras. Matrix coefficients of these intertwining operators are very useful in quantum integrable models [ES]. The powerful instrument for the study of the algebra of intertwining operators is a recurrence relation on the structure constants of the algebra, known as ABRR equation, see [ABRR].

Tarasov and Varchenko [TV] found the symmetries of the algebra of intertwining operators, which originate from morphisms of Verma modules. They satisfy braid group relations and transform the weights by means of a shifted Weyl group action. These symmetries got the name 'dynamical Weyl group'. The theory of dynamical Weyl groups was generalized to the quantum groups setup in [EV].

The form of operators of the dynamical Weyl group is very close to the factorized expressions for the extremal projector and for Zhelobenko cocycles [Zh2]. However, the precise statements and the origin of such a relation are not clear. One of our goals is to clarify this relation.

In this paper we establish a family of symmetries in a wide class of Mickelsson algebras. They form a representation of the related braid group by automorphisms of the Mickelsson algebra $Z^n(\mathcal{A})$ and transform Cartan elements by means of the shifted Weyl group action. Each generating automorphism is a product of the Zhelobenko 'cocycle' map q_{α_i} and an automorphism T_i of the algebra \mathcal{A} , extending the action of the Weyl group in $U(\mathfrak{g})$ (or Lusztig automorphism of $U_q(\mathfrak{g})$). The main new point of our approach is the homomorphism property of Zhelobenko maps, that was not noticed before. Unfortunately, the proof of this fact is not short and requires calculations with the extremal projector.

The construction of automorphisms of Mickelsson algebra $Z^n(\mathcal{A})$ is quite general. In particular, it covers examples of Mickelsson algebras, related to reductions $\mathfrak{g}' \supset \mathfrak{g}$ of one reductive Lie algebra to another, and of the smash product $U(\mathfrak{g}) \ltimes S(V)$ of $U(\mathfrak{g})$ and the symmetric algebra of a $U(\mathfrak{g})$ -module V , where it becomes the dynamical Weyl group action after a specialization of Cartan elements of \mathfrak{g} . It can be applied for the construction of finite-dimensional representations of Yangians and of quantum affine algebras, see [KN1, KN2].

The paper is organized as follows. In Section 2 we collect a necessary information about the extremal projector and required extensions of $U(\mathfrak{g})$.

In Section 3 for a fixed contragredient Lie algebra \mathfrak{g} of finite growth we introduce a class of associative algebras \mathcal{A} , which we call \mathfrak{g} -admissible. They contain $U(\mathfrak{g})$ as a subalgebra and the adjoint action of \mathfrak{g} in \mathcal{A} has special properties. In particular, \mathcal{A} is

equivalent to a tensor product of $U(\mathfrak{g})$ and some subspace $\mathcal{V} \in \mathcal{A}$ as a \mathfrak{g} -module with respect to the adjoint action. Mickelsson algebras, related to admissible algebras, have distinguished properties. The crucial one is the existence of two distinguished spaces of generators z_v and z'_v , $v \in \mathcal{V}$.

Section 4 is an exposition of the 'Zhelobenko cocycle' [Zh2]. We present it with complete proofs in order to eliminate unnecessary restrictions, assumed in [Zh2]. The story starts with the map q_α , which relates universal Verma modules, attached to different maximal nilpotent subalgebras of \mathfrak{g} . The product of such operators over the system of positive roots maps vectors $v \in \mathcal{V}$ to the generators z'_v of Mickelsson algebras. This invariant description proves the cocycle conditions for the maps q_α .

Section 5 describes homomorphic properties of Zhelobenko maps. We prove first that the Zhelobenko map q_α establishes an isomorphism of a double coset algebra and the Mickelsson algebra. This implies that the compositions \tilde{q}_i of Zhelobenko maps with extensions of Weyl group automorphisms are automorphisms of the Mickelsson algebras, satisfying braid group relations.

In Sections 6 and 8 we calculate the images of generators of Mickelsson algebras and of standard modules over them with respect to \tilde{q}_i and show that the dynamical Weyl group is a particular case of our construction. Section 7 is devoted to the Mickelsson algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$, related to \mathfrak{n}_- -coinvariants of \mathcal{A} -modules.

Section 9 is a sketch of extensions of the constructions to quantum groups $U_\nu(\mathfrak{g})$. The new important detail here is that compositions of Zhelobenko maps q_α with Lusztig automorphisms coincide with the compositions of q_α with the adjoint action of Lusztig automorphisms, see Proposition 9.2 and Proposition 9.4. This allows to prove both homomorphism properties and braid group relations. We conclude with remarks about the range of assumptions on \mathfrak{g} -admissible algebras, used in the paper.

2. EXTREMAL PROJECTOR

In this section we review Zhelobenko's approach to extremal projector of Asherova-Smirnov-Tolstoy [AST]. The exposition follows [Zh] in main details.

2.1. Taylor extension of $U(\mathfrak{g})$. Let \mathfrak{g} be a contragredient Lie algebra of finite growth with symmetrizable Cartan matrix $a_{i,j}$, $i, j = 1, \dots, r$. Let

$$(2.1) \quad \mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}$$

be its Gauss decomposition, where \mathfrak{h} is a Cartan subalgebra, $\mathfrak{n} = \mathfrak{n}_+ \subset \mathfrak{b}$ and $\mathfrak{n}_- \subset \mathfrak{b}_-$ are nilradicals of two opposite Borel subalgebras $\mathfrak{b} = \mathfrak{b}_+$ and \mathfrak{b}_- . We use the notation Π for the system of simple positive roots; Δ_\pm and $\Delta = \Delta_+ \amalg \Delta_-$ for the systems of positive, negative and all roots; Δ_\pm^{re} and $\Delta^{re} = \Delta_+^{re} \amalg \Delta_-^{re}$ for the systems of positive, negative and all real roots. Let $(,)$ be the scalar product in \mathfrak{h}^* , such that $(\alpha_i, \alpha_j) = d_i a_{i,j} = d_j a_{j,i}$, for $\alpha_i, \alpha_j \in \Pi$ and $d_i \in \mathbb{N}$.

Denote by $Q \subset \mathfrak{h}^*$ the root lattice, $Q = \mathbb{Z} \cdot \Delta$, and put $Q_\pm = \mathbb{Z}_{\geq 0} \cdot \Delta_\pm$. For any $\mu \in Q^\pm$ we denote by $U(\mathfrak{n})_\mu$ and $U(\mathfrak{n}_-)_\mu$ the subspace of elements x of $U(\mathfrak{n})$ and $U(\mathfrak{n}_-)$, such that $[h, x] = \langle \mu, h \rangle x$. We accept the normalization of Chevalley generators $e_{\alpha_i} \in \mathfrak{n}$, $e_{-\alpha_i} = f_{\alpha_i} \in \mathfrak{n}_-$, and of coroots $h_{\alpha_i} = \alpha_i^\vee \in \mathfrak{h}$, where $\alpha_i \in \Pi$, such that

$$\begin{aligned} [e_{\alpha_i}, e_{-\alpha_j}] &= \delta_{i,j} h_{\alpha_i}, \\ [h_{\alpha_i}, e_{\pm \alpha_j}] &= \pm a_{i,j} e_{\pm \alpha_j}, \\ \text{ad}_{e_{\pm \alpha_i}}^{1-a_{i,j}} e_{\pm \alpha_j} &= 0 \end{aligned} \quad \text{if } i \neq j.$$

For any $\gamma \in \Delta$ we define a coroot $h_\gamma \in \mathfrak{h}$ by the rule

$$(\alpha, \alpha)h_\alpha + (\beta, \beta)h_\beta = (\alpha + \beta, \alpha + \beta)h_{\alpha+\beta} \quad \text{if } \alpha, \beta, \alpha + \beta \in \Delta_+.$$

Let W be the Weyl group of \mathfrak{g} . For any $w \in W$ we denote by $T_w : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ a lift of the map $w : \mathfrak{h} \rightarrow \mathfrak{h}$ to the automorphism of the algebra $U(\mathfrak{g})$, satisfying braid group relations $T_{ww'} = T_w T_{w'}$ if $l(ww') = l(w) + l(w')$, where $l(w)$ is the length of w . For instance, we may choose T_w , as in [L]. We accept a shortened notation T_i for automorphisms $T_{s_{\alpha_i}}$, where $\alpha_i \in \Pi$.

Denote by D the localization of the free commutative algebra $U(\mathfrak{h})$ with respect to the multiplicative set of denominators, generated by

$$\{h_\alpha + k \mid \alpha \in \Delta, k \in \mathbb{Z}\}.$$

To any $\mu \in \mathfrak{h}^*$ we associate an automorphism τ_μ of the algebra D . It is uniquely defined by the conditions

$$(2.2) \quad \tau_\mu(h) = h + \langle h, \mu \rangle \quad \text{for any } h \in \mathfrak{h}.$$

Denote by $U'(\mathfrak{g})$ the extension of $U(\mathfrak{g})$ by means of D :

$$U'(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} D \approx D \otimes_{U(\mathfrak{h})} U(\mathfrak{g}).$$

Note that $U'(\mathfrak{g})$ is a D -bimodule and any automorphism T_w admits a canonical extension to an automorphism of $U'(\mathfrak{g})$, which we denote by the same symbol.

Choose a normal ordering (see [T] for the definition) $\gamma_1 \prec \gamma_2 \prec \dots \prec \gamma_n$ of the system Δ_+ of positive roots of \mathfrak{g} ($n = |\Delta_+|$ may be infinite). Let $e_{\pm\alpha}$, and h_α , where $\alpha \in \Delta$ be Cartan-Weyl generators, constructed by recursive procedure, attached to this order. We assume that they are normalized in such a way that

$$(2.3) \quad [e_\alpha, e_{-\alpha}] = h_\alpha, \quad [h_\alpha, e_{\pm\beta}] = \pm \langle h_\alpha, \beta \rangle e_{\pm\beta}.$$

For any $\bar{k} \in \mathbb{Z}_{\geq 0}^n$, $\bar{k} = (k_1, \dots, k_n)$ with $\sum k_i < \infty$ denote by $e_{\bar{k}}^+$ the monomial $e_{\bar{k}}^+ = e_{\gamma_1}^{k_1} \dots e_{\gamma_n}^{k_n} \in U(\mathfrak{b}_+)$ and by $e_{\bar{k}}^-$ the monomial $e_{\bar{k}}^- = e_{-\gamma_1}^{k_1} \dots e_{-\gamma_n}^{k_n} \in U(\mathfrak{b}_-)$. For every $\mu \in Q$ denote by $(F_{\mathfrak{g}, \mathfrak{n}})_\mu$ the vector space of series

$$(2.4) \quad x_\mu = \sum_{\bar{k}, \bar{r} \in \mathbb{Z}_{\geq 0}^n} e_{\bar{k}}^- x_{\bar{k}, \bar{r}} e_{\bar{r}}^+, \quad x_{\bar{k}, \bar{r}} \in D$$

of the total weight μ . Set

$$F_{\mathfrak{g}, \mathfrak{n}} = \bigoplus_{\mu \in Q} (F_{\mathfrak{g}, \mathfrak{n}})_\mu.$$

Proposition 2.1. (See [Zh], Section 3.2.3) *The space $F_{\mathfrak{g}, \mathfrak{n}}$ is an associative algebra with respect to the multiplication of formal series. Its definition does not depend on a choice of the normal ordering \prec .*

Clearly, $F_{\mathfrak{g}, \mathfrak{n}}$ contains $U'(\mathfrak{g})$ as a subalgebra. We call $F_{\mathfrak{g}, \mathfrak{n}}$ a *Taylor extension* of $U'(\mathfrak{g})$, related to the decomposition (2.1).

A choice of the normal ordering is a technical tool for a description of the algebra. It is used for a construction of particular bases in weight components of the algebras $U(\mathfrak{n}_\pm)$. Instead, one can fix, for any $\nu \in Q_+$, a basis $e_{\nu, j}^+$ of the finite-dimensional space $U(\mathfrak{n}_+)_\nu$ and, for any $\nu \in Q_-$, a basis $e_{\nu, j}^-$ of the finite-dimensional space $U(\mathfrak{n}_-)_\nu$. Then the space $F_{\mu, \mathfrak{g}}$ consists of formal series

$$x_\mu = \sum_{\nu \in Q_+, \nu' \in Q_-, j, j'} e_{\nu', j'}^- x_{\nu', j', \nu, j} e_{\nu, j}^+, \quad x_{\nu', j', \nu, j} \in D$$

of the total weight $\mu \in Q$.

2.2. Universal Verma module and extremal projector. Set

$$M_{\mathfrak{n}}(\mathfrak{g}) = U'(\mathfrak{g})/U'(\mathfrak{g})\mathfrak{n}.$$

The space $M_{\mathfrak{n}}(\mathfrak{g})$ is a left $U'(\mathfrak{g})$ -module and a D -bimodule. It is called the *universal Verma module*. Since $M_{\mathfrak{n}}(\mathfrak{g})$ is a $U(\mathfrak{h})$ -bimodule, we have an adjoint action of $U(\mathfrak{h})$ in $M_{\mathfrak{n}}(\mathfrak{g})$, defined as $ad_h(m) = [h, m]$ for any $h \in \mathfrak{h}$ and $m \in M_{\mathfrak{n}}(\mathfrak{g})$. We have the weight decomposition of $M_{\mathfrak{n}}(\mathfrak{g})$ with respect to the adjoint action of $U(\mathfrak{h})$:

$$M_{\mathfrak{n}}(\mathfrak{g}) = \bigoplus_{\mu \in Q_-} (M_{\mathfrak{n}}(\mathfrak{g}))_{\mu}.$$

Denote by $E'(\mathfrak{g})$ the algebra of all endomorphisms of $M_{\mathfrak{n}}(\mathfrak{g})$, which commute with the right action of $U(\mathfrak{h})$. We have a linear map $\xi : U'(\mathfrak{g}) \rightarrow E'(\mathfrak{g})$, induced by the multiplication in $U'(\mathfrak{g})$ and establishing in $M_{\mathfrak{n}}(\mathfrak{g})$ the structure of the left $U'(\mathfrak{g})$ -module. For any $\mu \in Q$, define

$$E_{\mu}(\mathfrak{g}) = \{a \in E'(\mathfrak{g}) \mid [\xi(h), a] = \langle \mu, h \rangle a \text{ for any } h \in \mathfrak{h}\},$$

and set

$$E(\mathfrak{g}) = \bigoplus_{\mu \in Q} E_{\mu}(\mathfrak{g}).$$

Proposition 2.2. (See [Zh], Sections 3.2.4.-3.2.5) *The map ξ induces an isomorphism of algebras:*

$$(2.5) \quad \xi : F_{\mathfrak{g}, \mathfrak{n}} \rightarrow E(\mathfrak{g}).$$

The proof of Proposition 2.2 is strongly based on the nondegeneracy of Shapovalov form.

Recall [Sh] that the Shapovalov form $A(x, y) : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \rightarrow U(\mathfrak{h})$ is defined by the relation $A(x, y) = \beta(x^t \cdot y)$, where $x \mapsto x^t$ is the Chevalley antiinvolution ($e_{\pm\alpha}^t = e_{\mp\alpha}$, $h^t = h$, $(xy)^t = y^t x^t$) in $U(\mathfrak{g})$ and $\beta : U(\mathfrak{g}) \rightarrow U(\mathfrak{h})$ is the projection with respect to the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n}).$$

The Shapovalov form vanishes on the left ideal $U(\mathfrak{g})\mathfrak{n}$ and is nondegenerate on $U(\mathfrak{n}_-)$. It admits an extension to a nondegenerate form on $M_{\mathfrak{n}}(\mathfrak{g})$ with values in D .

For the proof of the isomorphism (2.5) we choose for any $\nu \in Q^-$ a basis $e_{\nu, j}^-$ of $U(\mathfrak{n}_-)_{\nu}$, which is orthogonal with respect to the Shapovalov form, and take $e_{-\nu, j}^+ = (e_{\nu, j}^-)^t$.

Proposition 2.3. (See [Zh], Section 3.2.8) *There exists a unique element $P_{\mathfrak{n}} \in F_{\mathfrak{g}, \mathfrak{n}}$, satisfying equations*

$$(2.6) \quad e_{\alpha} P_{\mathfrak{n}} = P_{\mathfrak{n}} e_{-\alpha} = 0 \quad \text{for all } \alpha \in \Delta_+$$

with zero term $\beta(P_{\mathfrak{n}}) = 1$. It is a self-adjoint projector of zero weight, $P_{\mathfrak{n}}^2 = P_{\mathfrak{n}}$, $P_{\mathfrak{n}}^t = P_{\mathfrak{n}}$.

The definition and the construction of the projector $P_{\mathfrak{n}}$ depends on a choice of the nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{g}$. When \mathfrak{n} coincides with a fixed nilpotent subalgebra \mathfrak{n} , entering the decomposition (2.1), we omit the label \mathfrak{n} and denote the projector simply by P .

Note an important property of P , which follows from Proposition 2.3:

$$(2.7) \quad P - 1 \in F_{\mathfrak{g}, \mathfrak{n}} \bigcap \mathfrak{n}_- F_{\mathfrak{g}, \mathfrak{n}} .$$

By means of Proposition 2.2, the element P is described as an element of $E(\mathfrak{g})$, which projects the universal Verma module $M_{\mathfrak{n}}(\mathfrak{g})$ to the subspace $(M_{\mathfrak{n}}(\mathfrak{g}))^{\mathfrak{n}} = M_{\mathfrak{n}}(\mathfrak{g})_0$ of \mathfrak{n} -invariants along $\mathfrak{n}_- M_{\mathfrak{n}}(\mathfrak{g}) = \bigoplus_{\gamma < 0} (M_{\mathfrak{n}}(\mathfrak{g}))_{\gamma}$. The element P is called the *extremal projector*. It was discovered in [AST].

There are three distinct cases of the use of the extremal projector.

1. Let V be a $F_{\mathfrak{g}, \mathfrak{n}}$ -module. Then P projects V on the subspace $V^{\mathfrak{n}}$ of \mathfrak{n} -invariants along the subspace $\mathfrak{n}_- V$.
2. Let V be a module over $U'(\mathfrak{g})$, locally finite with respect to \mathfrak{n} . Then P projects V on the subspace $V^{\mathfrak{n}}$ of \mathfrak{n} -invariants along the subspace $\mathfrak{n}_- V$.
3. Let V be a module over $U(\mathfrak{g})$, locally finite with respect to \mathfrak{b} . Assume that $\mu \in \mathfrak{h}^*$ satisfies the conditions:

$$(2.8) \quad \langle \mu + \rho, h_{\alpha} \rangle \neq -1, -2, \dots \quad \text{for any } \alpha \in \Delta_+ ,$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$.

Denote by V_{μ} the generalized weight subspace of V of the weight μ . Then P projects V_{μ} on $V_{\mu} \cap V^{\mathfrak{n}}$ along $V_{\mu} \cap \mathfrak{n}_- V$.

In the following, for any left $U(\mathfrak{g})$ -module M , in which the action of the projector P is defined, we denote the corresponding element of $\text{End } M$ by p , and for any right $U(\mathfrak{g})$ -module N , in which the action of the projector P is defined, we denote the corresponding element of $\text{End } N$ by \bar{p} .

The operator p satisfies the relations

$$(2.9) \quad p(e_{-\gamma} m) = e_{\gamma} p(m) = 0 \quad \text{for any } \gamma \in \Delta_+, m \in M, \quad p^2 = p .$$

The operator \bar{p} satisfies the relations

$$(2.10) \quad \bar{p}(n e_{\gamma}) = \bar{p}(n) e_{-\gamma} = 0 \quad \text{for any } \gamma \in \Delta_+, n \in N \quad \bar{p}^2 = \bar{p} .$$

2.3. Multiplicative formula for extremal projector. The extremal projector P for simple Lie algebras was discovered and investigated by Asherova, Smirnov and Tolstoy [AST]. They presented a multiplicative expression for P , which was later generalized to affine Lie superalgebras and their q -analogs. We reproduce here the formula of [AST].

For any $\alpha \in \Delta_+$ and $\lambda \in \mathfrak{h}^*$ let $f_{\alpha, n}[\lambda]$ and $g_{\alpha, n}[\lambda]$ be the following elements of D :

$$(2.11) \quad f_{\alpha, n}[\lambda] = \prod_{j=1}^n (h_{\alpha} + \langle h_{\alpha}, \lambda \rangle + j)^{-1}, \quad g_{\alpha, n}[\lambda] = \prod_{j=1}^n (-h_{\alpha} + \langle h_{\alpha}, \lambda \rangle + j)^{-1} .$$

Define $P_{\alpha}[\lambda] \in F_{\mathfrak{g}, \mathfrak{n}}$ and $P_{-\alpha}[\lambda] \in F_{\mathfrak{g}, \mathfrak{n}_-}$ by relations

$$(2.12) \quad P_{\alpha}[\lambda] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} f_{\alpha, n}[\lambda] e_{-\alpha}^n e_{\alpha}^n, \quad P_{-\alpha}[\lambda] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} g_{\alpha, n}[\lambda] e_{\alpha}^n e_{-\alpha}^n$$

Set

$$(2.13) \quad P_{\alpha} = P_{\alpha}[\rho], \quad P_{-\alpha} = P_{-\alpha}[\rho] .$$

Proposition 2.4. (See [AST].) *Let $\gamma_1 \prec \dots \prec \gamma_n$ be a normal ordering of Δ_+ . Then the extremal projector P is equal to the product*

$$(2.14) \quad P = \prod_{\gamma \in \Delta_+}^{\prec} P_\gamma,$$

where the order in the product coincides with the chosen normal order \prec .

Analogously, the product $\prod_{\gamma \in \Delta_+}^{\prec} P_{-\gamma}$ is equal in $F_{\mathfrak{g}, \mathfrak{n}_-}$ to the projector $P_{\mathfrak{n}_-}$.

For a generalization of (2.14) to arbitrary contragredient Kac-Moody Lie algebras of finite growth and their q -analogs, see [T, KT].

We define also elements $P[\lambda]$ and $P_-[\lambda]$ for any $\lambda \in \mathfrak{h}^*$ by the relations

$$(2.15) \quad P[\lambda] = \prod_{\gamma \in \Delta_+}^{\prec} P_\gamma[\lambda], \quad P_-[\lambda] = \prod_{\gamma \in \Delta_+}^{\prec} P_{-\gamma}[\lambda]$$

It is known (see [Zh], and also Section 3.5 of the present paper), that $P[\lambda]$ and $P_-[\lambda]$ do not depend on a choice of the normal order. In this notation, $P = P[\rho]$ and $P_{\mathfrak{n}_-} = P_-[\rho]$.

3. MICKELSSON ALGEBRAS

3.1. \mathfrak{g} -admissible algebras. Let \mathcal{A} be an associative algebra, which contains $U(\mathfrak{g})$ as a subalgebra. Then \mathcal{A} has a natural structure of a $U(\mathfrak{g})$ -bimodule. Since $U(\mathfrak{g})$ is a Hopf algebra, the bimodule structure produces an adjoint action of $U(\mathfrak{g})$ in \mathcal{A} . We have

$$(3.1) \quad \text{ad}_g(x) = \sum_i g'_i x S(g''_i),$$

where the coproduct $\Delta(\mathfrak{g})$ of the element $g \in U(\mathfrak{g})$ has a form $\Delta(g) = \sum_i g'_i \otimes g''_i$ and $S(g)$ is the antipode of g . The adjoint action ad_g of an element $g \in \mathfrak{g}$ is the commutator, $\text{ad}_g(a) = ga - ag$.

In the sequel we use the following notation for the adjoint action:

$$\hat{g}(a) \equiv \text{ad}_g(a), \quad g \in U(\mathfrak{g}), a \in \mathcal{A}.$$

We call \mathcal{A} a *\mathfrak{g} -admissible algebra*, if

- (a) there is a subspace $\mathcal{V} \subset \mathcal{A}$, invariant with respect to the adjoint action of $U(\mathfrak{g})$, such that the multiplication m in \mathcal{A} induces the following isomorphisms of vector spaces

$$(a1) \quad m : U(\mathfrak{g}) \otimes \mathcal{V} \rightarrow \mathcal{A}, \quad (a2) \quad m : \mathcal{V} \otimes U(\mathfrak{g}) \rightarrow \mathcal{A};$$

- (b) the adjoint action of real root vectors $e_\gamma \in U(\mathfrak{g})$ in \mathcal{V} is locally nilpotent. The adjoint action of the Cartan subalgebra \mathfrak{h} in \mathcal{V} is semisimple.

Sometimes we call \mathcal{V} an *ad-invariant generating subspace* of the \mathfrak{g} -admissible algebra \mathcal{A} .

The condition (a) says, in particular, that \mathcal{A} is a free left $U(\mathfrak{h})$ -module and a free right $U(\mathfrak{h})$ -module.

Since the adjoint action of real root vectors in $U(\mathfrak{g})$ is locally nilpotent, the conditions (a) and (b) imply that the adjoint action of real root vectors e_γ is locally nilpotent in \mathcal{A} , that is, for any $a \in \mathcal{A}$ and $\gamma \in \Delta^{re}$ vectors $\text{ad}_{e_\gamma}^n(a)$ are zero for sufficiently big n .

Thus the restriction of the adjoint action of $U(\mathfrak{g})$ to any \mathfrak{sl}_2 -subalgebra, generated by real root vectors $e_{\pm\gamma}$, where $\gamma \in \Delta_+^{re}$, is locally finite in \mathcal{A} . Since the adjoint action of the Cartan subalgebra $U(\mathfrak{h})$ is semisimple, \mathcal{A} admits the weight decomposition with respect to the adjoint action of $U(\mathfrak{h})$.

There are two main classes of \mathfrak{g} -admissible algebras.

- 1 . Let \mathfrak{g} be a reductive finite-dimensional Lie algebra and \mathfrak{g}_1 a contragredient Lie algebra of finite growth, which contains \mathfrak{g} . Then the adjoint action of \mathfrak{g} in \mathfrak{g}_1 is locally finite and $\mathcal{A} = U(\mathfrak{g}_1)$ is a \mathfrak{g} -admissible algebra.
- 2 . Let \mathcal{V} be a $U(\mathfrak{g})$ -module algebra with a locally nilpotent action of real root vectors. This means that \mathcal{V} is an associative algebra, equipped with a structure of a $U(\mathfrak{g})$ -module, such that the action of real root vectors is locally finite. These two structures are related: the action of Lie generators $g \in \mathfrak{g}$ satisfy the Leibniz rule:

$$(3.2) \quad g(v_1 v_2) = g(v_1)v_2 + v_1 g(v_2).$$

Denote by $U(\mathfrak{g}) \ltimes \mathcal{V}$ the smash product of $U(\mathfrak{g})$ and \mathcal{V} . It is an associative algebra, generated by elements $g \in U(\mathfrak{g})$ and $v \in \mathcal{V}$, satisfying the relation

$$(3.3) \quad gv - vg = g(v), \quad g \in \mathfrak{g}, \quad v \in \mathcal{V},$$

and, more generally,

$$(3.4) \quad \sum_i g'_i v S(g''_i) = g(v), \quad g \in U(\mathfrak{g}), \quad v \in \mathcal{V},$$

where S is the antipode in $U(\mathfrak{g})$, $\Delta(g) = \sum_i g'_i \otimes g''_i$ is the comultiplication in $U(\mathfrak{g})$. The smash product $U(\mathfrak{g}) \ltimes \mathcal{V}$ is a \mathfrak{g} -admissible algebra.

- 2a . Let V be a $U(\mathfrak{g})$ -module and $\text{End}^0 V$ be the algebra of the endomorphisms of V , finite with respect to the adjoint action of real root vectors of $U(\mathfrak{g})$. Then the tensor product $U(\mathfrak{g}) \otimes \text{End}^0 V$ is a \mathfrak{g} -admissible algebra. This construction is a particular case of the previous one: the tensor product $U(\mathfrak{g}) \otimes \text{End}^0 V$ is a smash product of $\mathbb{C} \otimes \text{End}^0 V$ and of diagonally imbedded $U(\mathfrak{g})$, generated by the elements $g \otimes 1 + 1 \otimes g$, $g \in \mathfrak{g}$.

3.2. Mickelsson algebras. Definitions. Let \mathcal{A} be an associative algebra, which contains $U(\mathfrak{g})$. Let $\text{Nr}(\mathcal{A}\mathfrak{n})$ be the normalizer of the left ideal $\mathcal{A}\mathfrak{n}$:

$$x \in \text{Nr}(\mathcal{A}\mathfrak{n}) \equiv \mathfrak{n}x \subset \mathcal{A}\mathfrak{n}.$$

Denote by $S^n(\mathcal{A})$ the quotient space

$$S^n(\mathcal{A}) = \text{Nr}(\mathcal{A}\mathfrak{n}) / \mathcal{A}\mathfrak{n}.$$

Proposition 3.1.

- (i) The space $S^n(\mathcal{A})$ is an algebra with respect to the multiplication in \mathcal{A} ;
- (ii) Let M be an \mathcal{A} -module. Then the space M^n of \mathfrak{n} -invariant vectors in M is a $S^n(\mathcal{A})$ -module.

The algebra $S^n(\mathcal{A})$ is called *Mickelsson algebra* [M]. Since \mathfrak{h} normalizes \mathfrak{n}_{\pm} , we have the inclusion $U(\mathfrak{h}) \subset S^n(\mathcal{A})$. Denote by \mathcal{A}' the localization

$$\mathcal{A}' = D \otimes_{U(\mathfrak{h})} \mathcal{A}.$$

By the condition (a) of a \mathfrak{g} -admissible algebra, we have a canonical imbedding of \mathcal{A} into \mathcal{A}' and thus an adjoint action of $U(\mathfrak{g})$ in \mathcal{A}' , compatible with the adjoint action of $U(\mathfrak{g})$ in \mathcal{A} .

Define *Mickelsson algebra* $Z^n(\mathcal{A})$ as the quotient

$$Z^n(\mathcal{A}) = \text{Nr}(\mathcal{A}'\mathfrak{n})/\mathcal{A}'\mathfrak{n},$$

where $\text{Nr}(\mathcal{A}'\mathfrak{n})$ is the normalizer of the left ideal $\mathcal{A}'\mathfrak{n}$ of \mathcal{A}' . The algebra $Z^n(\mathcal{A})$ is a localization of the algebras $S^n(\mathcal{A})$:

$$Z^n(\mathcal{A}) = D \otimes_{U(\mathfrak{h})} S^n(\mathcal{A}).$$

We can change the order of taking quotients and subspaces in the definition of Mickelsson algebra. Then the Mickelsson algebra $Z^n(\mathcal{A})$ is defined as a subspace of \mathfrak{n} -invariants in a left $U(\mathfrak{g})$ -module $M_n(\mathcal{A}') = \mathcal{A}'/\mathcal{A}'\mathfrak{n}$:

$$Z^n(\mathcal{A}) = (M_n(\mathcal{A}'))^{\mathfrak{n}} = \{m \in M_n(\mathcal{A}') \mid \mathfrak{n}m = 0\}.$$

The algebra Z^n acts in the space M^n of \mathfrak{n} -invariants of any \mathcal{A}' -module M .

3.3. Double coset algebra. Suppose that a \mathfrak{g} -admissible algebra \mathcal{A} satisfies the additional *local highest weight* condition:

(HW) For any $v \in \mathcal{V}$, the adjoint action of elements $x \in U(\mathfrak{n})_\mu$ on v is nontrivial, $\hat{x}(v) \neq 0$, only for a finite number of $\mu \in \mathfrak{h}^*$.

With this assumption the quotient $M_n(\mathcal{A}') = \mathcal{A}'/\mathcal{A}'\mathfrak{n}$ has a structure of a left $F_{\mathfrak{g},\mathfrak{n}}$ -module, extending the action of \mathcal{A}' by left multiplication. In particular, the extremal projector P acts in the left $F_{\mathfrak{g},\mathfrak{n}}$ -module $M_n(\mathcal{A}')$.

The properties of the extremal projectors imply the relation

$$(3.5) \quad Z^n(\mathcal{A}) = \text{Im } p \subset M_n(\mathcal{A}'),$$

where $p \in \text{End } M_n(\mathcal{A}')$ is the action of P in $M_n(\mathcal{A}')$, see Section 2.2.

Denote by ${}_n\mathcal{A}_n$ the double coset space

$$(3.6) \quad {}_n\mathcal{A}_n = \mathfrak{n}_-\mathcal{A}' \backslash \mathcal{A}'/\mathcal{A}'\mathfrak{n} \equiv \mathcal{A}'/(\mathfrak{n}_-\mathcal{A}' + \mathcal{A}'\mathfrak{n}).$$

Equip ${}_n\mathcal{A}_n$ with a binary operation $\circ : {}_n\mathcal{A}_n \otimes {}_n\mathcal{A}_n \rightarrow {}_n\mathcal{A}_n$:

$$(3.7) \quad a \circ b = aPb \stackrel{\text{def}}{=} a \cdot p(b).$$

The rule (3.7) means the following. For a class \bar{x} in ${}_n\mathcal{A}_n$, we take its representative $x \in \mathcal{A}'$. For a class \bar{y} in ${}_n\mathcal{A}_n$, we take its representative $y \in M_n(\mathcal{A}')$. Consider an element $xP(y)$ in $M_n(\mathcal{A}') = \mathcal{A}'/\mathcal{A}'\mathfrak{n}$. Then its class modulo $\mathfrak{n}_-M_n(\mathcal{A}')$ defines an element $\bar{x} \circ \bar{y}$ of ${}_n\mathcal{A}_n$. It does not depend on a choice of representatives.

We call the double coset space ${}_n\mathcal{A}_n$, equipped with the operation (3.7) the *double coset algebra* ${}_n\mathcal{A}_n$.

Define linear maps $\phi^+ : Z^n(\mathcal{A}) \rightarrow {}_n\mathcal{A}_n$ and $\psi^+ : {}_n\mathcal{A}_n \rightarrow Z^n(\mathcal{A})$ by the rules

$$(3.8) \quad \phi^+(x) = x \mod \mathfrak{n}_-M_n(\mathcal{A}'), \quad \psi^+(y) = p(y), \quad x \in Z^n(\mathcal{A}), y \in {}_n\mathcal{A}_n.$$

Let us explain the formula $\psi^+(y) = p(y)$. For a class y in ${}_n\mathcal{A}_n = \mathcal{A}'/\mathfrak{n}_-\mathcal{A}' + \mathcal{A}'\mathfrak{n}$, we choose its representative $\bar{y} \in M_n(\mathcal{A}') = \mathcal{A}'/\mathcal{A}'\mathfrak{n}$ and take $p(\bar{y})$. The result does not depend on a choice of a representative and is denoted by $\psi^+(y)$.

Proposition 3.2. *Assume that a \mathfrak{g} -admissible algebra \mathcal{A} satisfies the condition (HW). Then*

- (i) The operation (3.7) equips ${}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n}}$ with a structure of an associative algebra.
- (ii) The linear maps ϕ^+ and ψ^+ are inverse to each other and establish an isomorphism of algebras $Z^n(\mathcal{A})$ and ${}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n}}$.

Proof. Let $x \in Z^n(\mathcal{A}')$. Then $p(x) = x \bmod \mathcal{A}'\mathbf{n}$ due to (2.7). Thus $\psi^+ \cdot \phi^+ = Id_{Z^n(\mathcal{A})}$. On the other hand, due to the same property of P , for any $y \in \mathcal{A}'/\mathcal{A}'\mathbf{n}$,

$$p(y) = y \bmod \mathbf{n}_-\mathcal{A}' + \mathcal{A}'\mathbf{n}.$$

Thus $\phi^+ \cdot \psi^+ = Id_{{}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n}}}$. So the maps ϕ^+ and ψ^+ are inverse to each other.

Let now $x \in \mathcal{A}'$ and $y \in \mathcal{A}'$ be representatives of classes \tilde{x} and \tilde{y} in ${}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n}}$, $\bar{x} = x \bmod \mathcal{A}'\mathbf{n}$ and $\bar{y} = y \bmod \mathcal{A}'\mathbf{n}$ be their images in $M_{\mathbf{n}}(\mathcal{A}')$. We have $\psi^+(\tilde{x}) = p(\bar{x})$, $\psi^+(\tilde{y}) = p(\bar{y})$ and $\psi^+(\tilde{x} \circ \tilde{y}) = p(x \cdot p(\bar{y}))$.

On the other hand, the multiplication rule m in $Z^n(\mathcal{A})$ can be written as follows. Let $z, u \in Z^n(\mathcal{A}')$. Let $z' \in \text{Nr}(\mathcal{A}'\mathbf{n})$ be a representative of a class $z \in \mathcal{A}'/\mathcal{A}'\mathbf{n}$. Then $m(z \otimes u) = z' \cdot u$ as an element of $M_{\mathbf{n}}(\mathcal{A}')$. By (3.5), $m(z, u) = p(z' \cdot u)$. Thus we have in $Z^n(\mathcal{A}')$

$$m(p(x) \otimes p(y)) = p(p(x)' \cdot p(y)),$$

where $p(x)'$ is a representative of a class $p(x)$ in \mathcal{A}' . By the property (2.7),

$$p(x)' = x + x' + x'',$$

where $x' \in \mathbf{n}_-\mathcal{A}'$ and $x'' \in \mathcal{A}'\mathbf{n}$. Thus

$$p(p(x)' \cdot p(y)) = p(x \cdot p(y))$$

due to (2.7) and (2.9). Thus ψ^+ is a homomorphism, which proves simultaneously (i) and (ii). \square

3.4. Generators of Mickelsson algebras. Let \mathcal{A} be a \mathfrak{g} -admissible algebra with an ad-invariant generating subspace \mathcal{V} , satisfying the highest weight condition (HW). By the condition (a) of a \mathfrak{g} -admissible algebra (see Section 3.1) and the PBW theorem for the algebra $U(\mathfrak{g})$, any element of \mathcal{A}' can be uniquely presented in the following form

$$x = \sum_i f_i d_i e_i v_i, \quad \text{where } f_i \in U(\mathbf{n}_-), d_i \in D, e_i \in U(\mathbf{n}), v_i \in \mathcal{V}.$$

Due to the highest weight condition (HW), we can move all e_i to the right and get a presentation

$$x = \sum_i f'_i d'_i v'_i e'_i, \quad \text{where } f'_i \in U(\mathbf{n}_-), d'_i \in D, e'_i \in U(\mathbf{n}), v'_i \in \mathcal{V}.$$

In the double coset space this presentation gives

$$x = \sum_i d'_i v'_i, \quad \bmod \mathbf{n}_-\mathcal{A}' + \mathcal{A}'\mathbf{n}, \quad \text{where } d'_i \in D, v'_i \in \mathcal{V}.$$

Proposition 3.3. *Let \mathcal{A} be a \mathfrak{g} -admissible algebra satisfying the highest weight condition (HW). Then*

- (i) *Each element of the double coset algebra ${}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n}}$ can be uniquely presented in a form $x = \sum_i d_i v_i$, where $d_i \in D$, $v_i \in \mathcal{V}$, so that ${}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n}}$ is a free left (and right) D module, isomorphic to $D \otimes \mathcal{V}$ ($\mathcal{V} \otimes D$).*
- (ii) *For each $v \in \mathcal{V}$ there exists a unique element $z_v \in Z^n(\mathcal{A})$ of the form*

$$(3.9) \quad z_v = v + \sum_{i=1} d_i f_i v_i, \quad f_i \in \mathbf{n}_-U(\mathbf{n}_-), d_i \in D, v_i \in \mathcal{V}$$

such that the algebra $Z^n(\mathcal{A})$ is a free left (and right) D -module, generated by the elements z_v . The element z_v is equal to $p(v)$.

Proof. The part (i) is already proved. Applying Proposition 3.2, we see that any element of $Z^n(\mathcal{A})$ can be presented in a form $\sum_i d_i p(w_i)$, where $d_i \in D$, $w_i \in \mathcal{V}$. The element $z_v = p(v)$ has a form (3.9) due to the definition of the operator p and is uniquely characterized by this presentation. \square

The Mickelsson algebras have distinguished generators of another type. Their existence is imposed by the following proposition.

Let \mathcal{A} be an arbitrary \mathfrak{g} -admissible algebra with an ad-invariant generating subspace \mathcal{V} .

Proposition 3.4. *For each $v \in \mathcal{V}$ there exists at most one element $z'_v \in Z^n(\mathcal{A})$ of the form*

$$(3.10) \quad z'_v = v + \sum_i d_i v_i f_i, \quad f_i \in \mathfrak{n}_- U(\mathfrak{n}_-), \quad d_i \in D, \quad v_i \in \mathcal{V}.$$

Proof. Consider the case of the algebra $Z^n(\mathcal{A})$. If $m \in M_n(\mathcal{A}') = \mathcal{A}'/\mathcal{A}'\mathfrak{n}$ is a highest weight vector, that is $e_\alpha m = 0$ for any $\alpha \in \Delta_+$, then dm is also a highest weight vector for any $d \in D$. Thus (i) is equivalent to the statement that for any $\gamma \in \mathfrak{h}^*$ there is no highest weight vector of the form

$$(3.11) \quad x = \sum_i d_i v_i f_i, \quad f_i \in \mathfrak{n}_- U(\mathfrak{n}_-), \quad d_i \in D, \quad v_i \in \mathcal{V},$$

where all the terms have the weight γ with respect to the adjoint representation of \mathfrak{h} . In other words, we should prove that the conditions $[e_{\alpha_i}, x] = 0$ in $M_n(\mathcal{A})$ imply $x = 0$ in $M_n(\mathcal{A})$ if all $f_j \in \mathfrak{n}_- U(\mathfrak{n}_-)$.

By the condition (a2) of a \mathfrak{g} -admissible algebra and the PBW theorem for $U(\mathfrak{g})$ the elements $v_i f_i$ form a basis of $M_n(\mathcal{A}')$ over D if v_i form a basis of \mathcal{V} and f_i form a basis of $U(\mathfrak{n}_-)$. Consider the terms of the right hand side of (3.11) with v_j having minimal weights with respect to other weights which occur in (3.11). Then the expression $[e_{\alpha_i}, x]$ contain terms $v_j [e_{\alpha_i}, f_j]$ which are nonzero for some α_i if $f_j \in \mathfrak{n}_- U(\mathfrak{n}_-)$. This is because all the highest weight vectors of $M_n(\mathfrak{g})$ have zero weight. Thus x cannot be a highest weight vector. \square

We now specify to a case when \mathcal{A} is an admissible algebra over a finite-dimensional reductive Lie algebra.

Theorem 1. *Let \mathfrak{g} be a finite-dimensional reductive Lie algebra and \mathcal{A} a \mathfrak{g} -admissible algebra with generating subspace \mathcal{V} . Then for any $v \in \mathcal{V}$ there exists a unique element $z'_v \in Z^n(\mathcal{A})$ (3.10). The algebra $Z^n(\mathcal{A})$ is generated by elements z'_v as a free left (and right) D -module.*

Proof of Theorem 1 will be given in the next Section.

3.5. Relations between two sets of generators. Extend the notation of canonical generators of Mickelsson algebras to the elements of $D \otimes_{\mathbb{C}} \mathcal{V}$ and $\mathcal{V} \otimes_{\mathbb{C}} D$. We set for any $d \in D$ and $v \in \mathcal{V}$

$$(3.12) \quad z_{d \otimes v} = d \cdot z_v, \quad z'_{d \otimes v} = d \cdot z'_v, \quad z_{v \otimes d} = z_v \cdot d, \quad z'_{v \otimes d} = z'_v \cdot d \quad \text{in} \quad Z^n(\mathcal{A}).$$

Fix a positive real root α . We define now certain operators in a vector space $\mathcal{V} \otimes D$. Accept the notation $A^{(1)}$ for the operator $A \otimes 1$ in a vector space $\mathcal{V} \otimes D$ and $A^{(2)}$ for the operator $1 \otimes A$.

Let α be a real root, For any $\mu \in \mathfrak{h}^*$ and $n \geq 0$ define operators $\bar{f}_{n,\alpha}^{(1)}[\mu]$, $\bar{g}_{n,\alpha}^{(1)}[\mu]$, $B_\alpha^{(1)}[\mu]$ and $C_{-\alpha}^{(1)}[\lambda] \in \text{End}(\mathcal{V} \otimes D)$ by the relations

$$(3.13) \quad \begin{aligned} \bar{f}_{n,\alpha}^{(1)}[\mu] &= \prod_{k=1}^n \left(\hat{h}_\alpha^{(1)} + h_\alpha^{(2)} + \langle h_\alpha, \mu \rangle + k \right)^{-1}, \\ \bar{g}_{n,\alpha}^{(1)}[\mu] &= \prod_{k=1}^n \left(-\hat{h}_\alpha^{(1)} - h_\alpha^{(2)} + \langle h_\alpha, \mu \rangle + k \right)^{-1}, \end{aligned}$$

$$(3.14) \quad \begin{aligned} B_\alpha^{(1)}[\mu] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \bar{f}_{n,\alpha}^{(1)}[\mu] (\hat{e}_{-\alpha}^{(1)})^n (\hat{e}_\alpha^{(1)})^n, \\ C_{-\alpha}^{(1)}[\mu] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \bar{g}_{n,\alpha}^{(1)}[\mu] (\hat{e}_\alpha^{(1)})^n (\hat{e}_{-\alpha}^{(1)})^n. \end{aligned}$$

Here $\hat{h}_\alpha^{(1)} = \text{ad}_{h_\alpha}^{(1)}$ is the adjoint action of h_α in \mathcal{V} , $h_\alpha^{(2)}$ is the operator of multiplication by h_α in D .

Define also operators $\bar{f}_{n,\alpha}^{(2)}[\mu]$, $\bar{g}_{n,\alpha}^{(2)}[\mu]$, $C_\alpha^{(2)}[\mu]$ and $B_{-\alpha}^{(2)}[\lambda] \in \text{End}(D \otimes)$ by the following relations

$$(3.15) \quad \begin{aligned} \bar{f}_{n,\alpha}^{(2)}[\mu] &= \prod_{k=1}^n \left(\hat{h}_\alpha^{(2)} - h_\alpha^{(1)} + \langle h_\alpha, \mu \rangle + k \right)^{-1}, \\ \bar{g}_{n,\alpha}^{(2)}[\mu] &= \prod_{k=1}^n \left(-\hat{h}_\alpha^{(2)} + h_\alpha^{(1)} + \langle h_\alpha, \mu \rangle + k \right)^{-1}, \end{aligned}$$

$$(3.16) \quad \begin{aligned} C_\alpha^{(2)}[\mu] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \bar{f}_{n,\alpha}^{(2)}[\mu] (\hat{e}_{-\alpha}^{(2)})^n (\hat{e}_\alpha^{(2)})^n, \\ B_{-\alpha}^{(2)}[\mu] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \bar{g}_{n,\alpha}^{(2)}[\mu] (\hat{e}_\alpha^{(2)})^n (\hat{e}_{-\alpha}^{(2)})^n. \end{aligned}$$

Here $\hat{h}_\alpha^{(2)} = \text{ad}_{h_\alpha}^{(2)}$ is the adjoint action of h_α in \mathcal{V} , $h_\alpha^{(1)}$ is the operator of multiplication by h_α in D .

Let \mathfrak{g} be a reductive finite-dimensional Lie algebra. Let \prec be a normal ordering of the system Δ_+ of positive roots. Set

$$\begin{aligned} B^{(1)}[\lambda] &= \prod_{\gamma \in \Delta_+}^{\prec} B_\gamma^{(1)}[\lambda], & C_-^{(1)}[\lambda] &= \prod_{\gamma \in \Delta_+}^{\prec} C_{-\gamma}^{(1)}[\lambda], \\ C^{(2)}[\lambda] &= \prod_{\gamma \in \Delta_+}^{\prec} C_\gamma^{(2)}[\lambda], & B_-^{(2)}[\lambda] &= \prod_{\gamma \in \Delta_+}^{\prec} B_{-\gamma}^{(2)}[\lambda]. \end{aligned}$$

Theorem 2. *Let \mathcal{A} be an admissible algebra with a generating subspace \mathcal{V} over a finite-dimensional reductive Lie algebra \mathfrak{g} . Then for any $v \in \mathcal{V}$ we have the following equality in $Z^n(\mathcal{A})$*

$$(3.17) \quad z_v = z'_{B^{(1)}[\rho](v \otimes 1)}.$$

In particular, operators $B^{(1)}[\rho](v) : \mathcal{V} \otimes D \rightarrow \mathcal{V} \otimes D$ do not depend on a choice of the normal order \prec .

Proof. Consider the left $F_{\mathfrak{g}, \mathfrak{n}}$ -module $M_{\mathfrak{n}}(\mathcal{A}) = \mathcal{A}'/\mathcal{A}'\mathfrak{n}$. The multiplication in \mathcal{A}' induces an isomorphism of vector spaces $M_{\mathfrak{n}}(\mathcal{A})$ and $\mathcal{V} \otimes M_{\mathfrak{n}}(\mathfrak{g})$, where $M_{\mathfrak{n}}(\mathfrak{g}) = U'(\mathfrak{g})/U'(\mathfrak{g})\mathfrak{n}$:

$$(3.18) \quad m : \mathcal{V} \otimes M_{\mathfrak{n}}(\mathfrak{g}) \rightarrow M_{\mathfrak{n}}(\mathcal{A}).$$

With this identification the tensor product $\mathcal{V} \otimes M_{\mathfrak{n}}(\mathfrak{g})$ becomes a $F_{\mathfrak{g}, \mathfrak{n}}$ -module. As a $U(\mathfrak{g})$ -module it coincides with the tensor product of \mathcal{V} , equipped with a structure of the adjoint representation of $U(\mathfrak{g})$, and of the left $U(\mathfrak{g})$ -module $M_{\mathfrak{n}}(\mathfrak{g}) = U'(\mathfrak{g})/U'(\mathfrak{g})\mathfrak{n}$. This follows from the Leibniz rule

$$g(v \cdot x) = (gv - vg) \cdot x + v \cdot (gx) = \hat{g}(v) \cdot x + v \cdot gx,$$

for any $g \in \mathfrak{g}$, $v \in \mathcal{V}$ and $x \in U'(\mathfrak{g})/U'(\mathfrak{g})\mathfrak{n}$.

The elements of D act by the following rule: for any $d \in D$, $v \in \mathcal{V}$ of the weight μ_v and $x \in M_{\mathfrak{n}}(\mathcal{A})$ we have $d \cdot (v \otimes x) = v \otimes \tau_{\mu_v}(d)x$. Due to local finiteness of the adjoint action in \mathcal{V} these prescriptions define correctly the action of $F_{\mathfrak{g}, \mathfrak{n}}$ in $\mathcal{V} \otimes M_{\mathfrak{n}}(\mathfrak{g})$.

Under the identification (3.18) we have

$$z_v = p(v \otimes 1).$$

In order to express z_v via z'_v , we should write it in a form $v'_i \otimes d_i$ + lower order terms, where 'lower order terms' contain vectors, whose second tensor component lies in $D \cdot \mathfrak{n}_- U(\mathfrak{n}_-)$. Write P as a series over ordered monomials in e_{γ_i} , and $e_{-\gamma_i}$, where $\gamma_i \in \Delta_+$, with coefficients being rational functions of h_{γ_i} such that in any monomial all $e_{-\gamma_i}$ stand before all e_{γ_j} , in accordance to the rules of $F_{\mathfrak{g}, \mathfrak{n}}$,

$$P = P(h_{\gamma_i}, e_{-\gamma_i}, e_{\gamma_i}).$$

By a coproduct rule, in the action of P in $\mathcal{V} \otimes M_{\mathfrak{n}}(\mathfrak{g})$ we substitute, instead of $e_{\pm\gamma_i}$, the sum $\hat{e}_{\pm\gamma_i}^{(1)} + e_{\pm\gamma_i}^{(2)}$, and, instead of h_{γ_i} , the sum $\hat{h}_{\gamma_i}^{(1)} + h_{\gamma_i}^{(2)}$, each term acting in the corresponding tensor component. The action of $e_{\gamma_i}^{(2)}$ on $v \otimes 1$ vanishes, the action of $e_{-\gamma_i}^{(2)}$ gives 'lower order terms'. So the term we are looking for is equal to

$$p \left(\left(\hat{h}_{\gamma_i}^{(1)} + h_{\gamma_i}^{(2)} \right), \hat{e}_{-\gamma_i}^{(1)}, \hat{e}_{\gamma_i}^{(1)} \right) (v \otimes 1).$$

This is precisely $B^{(1)}[\rho](v \otimes 1)$. □

The operators $B^{(1)}[\rho]$, $B_-^{(2)}[\rho]$, $C^{(2)}[-\rho]$ and $C_-^{(1)}[-\rho]$, are closely related to the operators $P[\lambda]$ and $P_-[\lambda]$, see (2.15). Namely, denote by $\rho^{(1)}$ and $\rho^{(2)}$ the expressions

$$\rho^{(1)} = \frac{1}{2} \sum_{\gamma \in \Delta_+} h_{\gamma} \otimes \gamma, \quad \rho^{(2)} = \frac{1}{2} \sum_{\gamma \in \Delta_+} \gamma \otimes h_{\gamma}.$$

Then

$$\begin{aligned} B^{(1)}[\rho] &= \hat{p}^{(1)}[(\rho + \rho^{(2)})], & C_-^{(1)}[-\rho] &= \hat{p}_-^{(1)}[-(\rho + \rho^{(2)})], \\ B_-^{(2)}[\rho] &= \hat{p}_-^{(2)}[(\rho + \rho^{(1)})], & C^{(2)}[-\rho] &= \hat{p}^{(2)}[-(\rho + \rho^{(1)})]. \end{aligned}$$

The changes of coordinates, described in Theorem 2 and Theorem 5 below can be inverted by means of the relation

$$(3.19) \quad P_{\alpha}[\lambda] P_{-\alpha}[-\lambda] = \frac{h_{\alpha} + \langle h_{\alpha}, \lambda \rangle}{\langle h_{\alpha}, \lambda \rangle}.$$

This relation makes sense for a generic λ in any finite-dimensional representation of \mathfrak{sl}_2 , see [TV], Theorem 10.

So we have

$$(3.20) \quad \begin{aligned} B^{(1)}[\rho]^{-1} &= C_-^{(1)}[-\rho] \prod_{\alpha \in \Delta_+} \frac{h_\alpha^{(2)} + \langle h_\alpha, \rho \rangle}{\hat{h}_\alpha^{(1)} + h_\alpha^{(2)} + \langle h_\alpha, \rho \rangle}, \\ B_-^{(2)}[\rho]^{-1} &= C_-^{(2)}[-\rho] \prod_{\alpha \in \Delta_+} \frac{h_\alpha^{(1)} + \langle h_\alpha, \rho \rangle}{h_\alpha^{(1)} - \hat{h}_\alpha^{(2)} + \langle h_\alpha, \rho \rangle}. \end{aligned}$$

Proof of Theorem 1. The inversion relations (3.20) imply that

$$(3.21) \quad z'_v = z_{\gamma_1 C_-^{(1)}[-\rho](1 \otimes v)},$$

where $\gamma_1 = \prod_{\alpha \in \Delta_+} \frac{h_\alpha^{(2)} + \langle h_\alpha, \rho \rangle}{\hat{h}_\alpha^{(1)} + h_\alpha^{(2)} + \langle h_\alpha, \rho \rangle}$. Thus we have correctly defined elements z'_v , which proves the first statement of the Theorem. Other statements follow from the corresponding statements of Proposition 3.3. \square

Remark. If we replace the ring D by the field of fractions $\tilde{D} = \text{Frac}(U(\mathfrak{h}))$, the statement of Theorem 1 does not require the precise inversion relation (3.21). We just note that all the elements v_i , entering the right hand side of (3.9), belong to a finite-dimensional ad-invariant subspace $V \subset \mathcal{V}$, generated by v . We move then all f_i to the right and get a sum of elements z'_{v_k} with coefficients in D . If we allow coefficients in \tilde{D} , such a transformation defines an injective operator in $\tilde{D} \otimes V$, which is a finite-dimensional vector space over \tilde{D} . Thus this operator is invertible. An advantage of the precise formula (3.21) is that it shows that the coefficients of the inverse matrix belong to D .

4. ZHELOBENKO MAPS

4.1. Maps $q_\alpha^{(k)}$ and q_α . Let α be a real root of \mathfrak{g} . For any $x \in \mathcal{A}$ and $k \geq 0$ denote by $q_\alpha^{(k)}(x)$ the following element of $\mathcal{A}'/\mathcal{A}'e_\alpha$:

$$(4.1) \quad q_\alpha^{(k)}(x) = \sum_{n=k}^{\infty} \frac{(-1)^n}{(n-k)!} \hat{e}_\alpha^{n-k}(x) \cdot e_{-\alpha}^n \cdot g_{n,\alpha} \quad \text{mod } \mathcal{A}'e_\alpha,$$

where

$$(4.2) \quad g_{n,\alpha} = (h_\alpha(h_\alpha - 1) \cdots (h_\alpha - n + 1))^{-1}.$$

The assignment (4.1) has the following properties [Zh]:

$$(4.3) \quad \begin{aligned} (i) \quad & q_\alpha^{(k)}(xe_{-\alpha}) = 0, \\ (ii) \quad & [h, q_\alpha^{(k)}(x)] = q_\alpha^{(k)}([h, x]), \quad h \in \mathfrak{h}, \\ (iii) \quad & q_\alpha^{(k)}(xh) = q_\alpha^{(k)}(x)(h + \langle h, \alpha \rangle), \quad h \in \mathfrak{h}, \\ (iv) \quad & e_\alpha q_\alpha^{(k)}(x) = q_\alpha^{(k)}(e_\alpha x) = -k q_\alpha^{(k-1)}(x). \end{aligned}$$

Note that the quotient $\mathcal{A}'/\mathcal{A}'e_\alpha$ admits left and right actions of D , so the commutator $[h, q_\alpha^{(k)}(x)]$ is well defined in (ii). The property (iii) in the notation (2.2) can be written

as

$$(4.4) \quad q_\alpha^{(k)}(xh) = q_\alpha^{(k)}(x)\tau_\alpha(h), \quad h \in U(\mathfrak{h}).$$

We extend the assignment (4.1) to the linear map $q_\alpha^{(k)} : \mathcal{A}' \rightarrow \mathcal{A}'/\mathcal{A}'e_\alpha$ by means of the relation (iii) and denote by q_α the linear map $q_\alpha^{(0)} : \mathcal{A}' \rightarrow \mathcal{A}'/\mathcal{A}'e_\alpha$,

$$(4.5) \quad q_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{e}_\alpha^n(x) \cdot e_{-\alpha}^n \cdot g_{n,\alpha} \quad \text{mod } \mathcal{A}'e_\alpha.$$

The relations (4.3), (i), (iv) show that the image of q_α belongs to the normalizer of $\mathcal{A}'e_\alpha$ in \mathcal{A}' and the ideals $e_\alpha\mathcal{A}'$ and $\mathcal{A}'e_{-\alpha}$ are in the kernel of q_α .

Consider the one-dimensional vector space $\mathbb{C}e_\alpha$ as an abelian Lie algebra $\mathfrak{n}_\alpha = \mathbb{C}e_\alpha$ and one-dimensional vector space $\mathbb{C}e_{-\alpha}$ as an abelian Lie algebra $\mathfrak{n}_{-\alpha} = \mathbb{C}e_{-\alpha}$. Following the notation of Section 3.2, denote by $Z^{\mathfrak{n}_\alpha}(\mathcal{A}') = \text{Nr}(\mathcal{A}'e_\alpha)/\mathcal{A}'e_\alpha$ the Mickelsson algebra, related to the reduction to the corresponding sl_2 -subalgebra.

Proposition 4.1. *The map q_α defines an isomorphism of vector spaces ${}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}} \equiv e_\alpha\mathcal{A}' \setminus \mathcal{A}'/\mathcal{A}'e_{-\alpha}$ and $Z^{\mathfrak{n}_\alpha}(\mathcal{A}')$, such that for any $x \in {}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}}$, and $d \in D$*

$$(4.6) \quad [d, q_\alpha(x)] = q_\alpha([d, x]), \quad q_\alpha(xd) = q_\alpha(x)\tau_\alpha(d).$$

Proof. First of all note that the properties (4.3), (i) and (iv), say that the map q_α vanishes on $\mathfrak{n}_\alpha\mathcal{A}' + \mathcal{A}'\mathfrak{n}_{-\alpha}$ and thus defines a map of ${}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}}$ to $\mathcal{A}'/\mathcal{A}'\mathfrak{n}_\alpha$. Its image belongs to $Z^{\mathfrak{n}_\alpha}(\mathcal{A})$ by (4.3), (iv).

Let \mathfrak{g}_α be the sl_2 subalgebra of \mathfrak{g} , generated by e_α , $e_{-\alpha}$ and h_α .

Since α is a real root, the adjoint action of \mathfrak{g}_α in \mathfrak{g} is locally finite and semisimple. So there is a decomposition $\mathfrak{g} = \mathfrak{g}_\alpha + \mathfrak{p}$, invariant with respect to the adjoint action of \mathfrak{g}_α . Poincare-Birkhoff-Witt theorem implies that the multiplication in $U(\mathfrak{g})$ defines an isomorphism of tensor products $U(\mathfrak{g}_\alpha) \otimes S(\mathfrak{p})$ and $S(\mathfrak{p}) \otimes U(\mathfrak{g}_\alpha)$ with $U(\mathfrak{g})$. Here $S(\mathfrak{p})$ is regarded as a subspace of $U(\mathfrak{g})$, which consists of symmetric noncommutative polynomials on \mathfrak{p} . The space $S(\mathfrak{p})$ is invariant with respect to the adjoint action of \mathfrak{g}_α . Thus $U(\mathfrak{g})$ is \mathfrak{g}_α -admissible. Since the adjoint representation of $U(\mathfrak{g}_\alpha)$ in $U(\mathfrak{g})$ is locally finite, and \mathcal{A} is a \mathfrak{g} -admissible algebra, it is \mathfrak{g}_α -admissible as well. Let $\mathcal{V}_\alpha \subset \mathcal{A}$ be the subspace of \mathcal{A} , invariant with respect to the adjoint action of \mathfrak{g}_α and the multiplication in \mathcal{A} induces an isomorphism of vector spaces $U(\mathfrak{g}_\alpha) \otimes \mathcal{V}_\alpha$ and \mathcal{A} .

The double coset space ${}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}}$ is a free D -module, generated by the vector space \mathcal{V}_α (see the formulas in Section 3.4 with a replacement of \mathfrak{n} by $\mathfrak{n}_{-\alpha}$). On the other hand, the Mickelsson algebra $Z^{\mathfrak{n}_\alpha}(\mathcal{A}')$ is also a free D -module, generated, in the notation of the Remark in Section 3.4, by the vectors $z'_{\mathfrak{n}_\alpha, v}$, $v \in \mathcal{V}_\alpha$, see Theorem 1. By the structure of the map q_α , see (4.1), we have

$$q_\alpha(v) = v + \sum v_i f_i d_i,$$

where $v_i \in \mathcal{V}_\alpha$, f_i is a polynomial on $e_{-\alpha}$ without a constant term, $d_i \in D$. In other words,

$$(4.7) \quad q_\alpha(v) = z'_{\mathfrak{n}_\alpha, v}.$$

The relations (4.7) and (4.4) prove the proposition. \square

Let us restrict the map q_α to the normalizer $\text{Nr}(\mathcal{A}'\mathfrak{n}_{-\alpha})$. Due to (4.3), (i), this restriction defines a map $q_\alpha|_{\text{Nr}(\mathcal{A}'\mathfrak{n}_{-\alpha})} : Z^{\mathfrak{n}_{-\alpha}}(\mathcal{A}) \rightarrow Z^{\mathfrak{n}_\alpha}(\mathcal{A})$. We have also a map in other direction, $q_{-\alpha}|_{\text{Nr}(\mathcal{A}'\mathfrak{n}_\alpha)} : Z^{\mathfrak{n}_\alpha}(\mathcal{A}) \rightarrow Z^{\mathfrak{n}_{-\alpha}}(\mathcal{A})$.

Proposition 4.2. *We have equalities*

$$(4.8) \quad \begin{aligned} q_{-\alpha} q_{\alpha}(x) &= (h_{\alpha} + 1)x(h_{\alpha} + 1)^{-1} && \text{for any } x \in Z^{n_{-\alpha}}(\mathcal{A}), \\ q_{\alpha} q_{-\alpha}(y) &= (h_{\alpha} + 1)^{-1}y(h_{\alpha} + 1) && \text{for any } y \in Z^{n_{\alpha}}(\mathcal{A}). \end{aligned}$$

In particular, the restriction of the map q_{α} to the normalizer $\text{Nr}(\mathcal{A}'e_{-\alpha})$ defines an isomorphism of the vector spaces $Z^{n_{-\alpha}}(\mathcal{A})$ and $Z^{n_{\alpha}}(\mathcal{A})$. The inverse is given by the formula

$$y \mapsto (h_{\alpha} + 1)^{-1}q_{-\alpha}(y)(h_{\alpha} + 1), \quad y \in Z^{n_{\alpha}}(\mathcal{A}).$$

Proof. Take $y \in Z^{n_{\alpha}}(\mathcal{A}')$. We have

$$\begin{aligned} q_{\alpha} q_{-\alpha}(y) &= q_{\alpha} \left(\sum_{n \geq 0} \frac{(-1)^n}{n!} \hat{e}_{-\alpha}^n(y) e_{\alpha}^n \cdot g_{n, -\alpha} \right) \\ &= q_{\alpha} \left(\sum_{n \geq 0} \frac{1}{n!} \hat{e}_{-\alpha}^n(y) e_{\alpha}^n \right) \cdot ((h_{\alpha} + 2)(h_{\alpha} + 3) \cdots (h_{\alpha} + n + 1))^{-1} \\ &= q_{\alpha} \left(\sum_{n \geq 0} \frac{(-1)^n}{n!} \hat{e}_{\alpha}^n \hat{e}_{-\alpha}^n(y) \right) \cdot ((h_{\alpha} + 2)(h_{\alpha} + 3) \cdots (h_{\alpha} + n + 1))^{-1} \\ &= \sum_{m, n \geq 0} \frac{(-1)^{n+m}}{n!m!} \hat{e}_{\alpha}^m \hat{e}_{\alpha}^n \hat{e}_{-\alpha}^n(y) \cdot e_{-\alpha}^m ((h_{\alpha} + 2) \cdots (h_{\alpha} + n + 1))^{-1} \cdot g_{m, \alpha}. \end{aligned}$$

Since $q_{\alpha} q_{-\alpha}(y) \in Z^{n_{\alpha}}(\mathcal{A})$, we can replace it by $p_{\alpha}(q_{\alpha} q_{-\alpha}(y))$, where P_{α} is the projection operator P , related to the sl_2 -algebra \mathfrak{g}_{α} , and p_{α} is the action of P_{α} in $\mathcal{A}'/\mathcal{A}'\mathfrak{n}_{\alpha}$. Since $p_{\alpha}(e_{-\alpha}z) = 0$ for any $z \in \mathcal{A}'/\mathcal{A}'\mathfrak{n}_{\alpha}$, we have

$$\begin{aligned} q_{\alpha} q_{-\alpha}(y) &= p_{\alpha} \left(\sum_{m, n \geq 0} \frac{(-1)^n}{n!m!} \hat{e}_{-\alpha}^m \hat{e}_{\alpha}^m \hat{e}_{\alpha}^n \hat{e}_{-\alpha}^n(y) ((h_{\alpha} + 2) \cdots (h_{\alpha} + n + 1))^{-1} \cdot g_{m, \alpha} \right) \\ &= p_{\alpha} \left(\sum_{m, n \geq 0} \frac{(-1)^n}{n!m!} \prod_{k=1}^m (\hat{h}_{\alpha} - h_{\alpha} + k - 1)^{-1} \prod_{k=1}^n (-\hat{h}_{\alpha} + h_{\alpha} + k + 1)^{-1} \hat{e}_{-\alpha}^m \hat{e}_{\alpha}^m \hat{e}_{\alpha}^n \hat{e}_{-\alpha}^n(y) \right). \end{aligned}$$

The expression inside brackets can be interpreted as

$$\hat{p}_{\alpha}^{(2)}[-h_{\alpha}^{(1)} \otimes \rho - \rho] \cdot \hat{p}_{-\alpha}^{(2)}[h_{\alpha}^{(1)} \otimes \rho + \rho](1 \otimes y)$$

in $D \otimes Z^{n_{\alpha}}(\mathcal{A})$. Due to (3.19), it is equal to $\frac{\hat{h}_{\alpha} - h_{\alpha}^{(1)} - 1}{-h_{\alpha}^{(1)} - 1}(1 \otimes y) = \frac{-\hat{h}_{\alpha} + h_{\alpha}^{(1)} + 1}{h_{\alpha}^{(1)} + 1}(1 \otimes y)$. It means that

$$\begin{aligned} q_{\alpha} q_{-\alpha}(y) &= p_{\alpha} \left(\frac{-\hat{h}_{\alpha} + h_{\alpha} + 1}{h_{\alpha} + 1} y \right) = p_{\alpha} ((h_{\alpha} + 1)^{-1} y (h_{\alpha} + 1)) \\ &= (h_{\alpha} + 1)^{-1} p_{\alpha}(y) (h_{\alpha} + 1) = (h_{\alpha} + 1)^{-1} y (h_{\alpha} + 1). \end{aligned}$$

In the last line we used again the relation $y = p_{\alpha}(y) \bmod \mathcal{A}'\mathfrak{n}_{\alpha}$ for any $y \in Z^{n_{\alpha}}(\mathcal{A})$. The second relation is proved in an analogous manner. \square

4.2. Maps $q_{\alpha, \mathfrak{m}}^{(k)}$ and $q_{\alpha, \mathfrak{m}}$. Let α be a real root of \mathfrak{g} , e_α the corresponding root vector with respect to the decomposition (2.1). Let \mathfrak{m} be a maximal nilpotent subalgebra of \mathfrak{g} , conjugated to \mathfrak{n} by means of an element of the Weyl group W , such that e_α is a simple positive root vector of \mathfrak{m} . Set $\mathfrak{m}_- = \mathfrak{m}^t$, where $x \mapsto x^t$ is the Chevalley antiinvolution in $U(\mathfrak{g})$, see Section 2.2. For any $x \in \mathcal{A}$ and $k \geq 0$ denote by $q_{\alpha, \mathfrak{m}}^{(k)}(x)$ the following element of $\mathcal{A}'/\mathcal{A}'\mathfrak{m}$:

$$(4.9) \quad q_{\alpha, \mathfrak{m}}^{(k)}(x) = \sum_{n=k}^{\infty} \frac{(-1)^n}{(n-k)!} \hat{e}_\alpha^{n-k}(x) \cdot e_{-\alpha}^n \cdot g_{n, \alpha} \quad \text{mod } \mathcal{A}'\mathfrak{m},$$

where $g_{n, \alpha}$ is given by the relation (4.2). In other words,

$$(4.10) \quad q_{\alpha, \mathfrak{m}}^{(k)}(x) = \pi_{m, \alpha} \cdot q_\alpha^{(k)}(x),$$

where $\pi_{m, \alpha} : \mathcal{A}/\mathcal{A}e_\alpha \rightarrow \mathcal{A}/\mathcal{A}\mathfrak{m}$ is the natural factorization map. Due to (4.10), the assignment (4.9) satisfies the relations (4.3) and admits an extension to the map $q_{\alpha, \mathfrak{m}}^{(k)} : \mathcal{A}' \rightarrow \mathcal{A}'/\mathcal{A}'\mathfrak{m}$, satisfying the relation

$$(4.11) \quad [d, q_{\alpha, \mathfrak{m}}^{(k)}(x)] = q_{\alpha, \mathfrak{m}}^{(k)}([d, x]), \quad q_{\alpha, \mathfrak{m}}^{(k)}(xd) = q_{\alpha, \mathfrak{m}}^{(k)}(x)\tau_\alpha(d),$$

for any $x \in \mathcal{A}'$ and $d \in D$. We denote by $q_{\alpha, \mathfrak{m}}^{(0)}$ the map $q_{\alpha, \mathfrak{m}}^{(0)} : \mathcal{A}' \rightarrow \mathcal{A}'/\mathcal{A}'\mathfrak{m}$.

For any element $w \in W$, where W is the Weyl group of \mathfrak{g} , and a maximal nilpotent subalgebra $\mathfrak{m} \subset \mathfrak{g}$, we denote by $\mathfrak{m}^w \subset \mathfrak{g}$ the nilpotent subalgebra $\mathfrak{m}^w = T_w(\mathfrak{m})$, see Section 2.1.

Lemma 4.3. *For any $k \geq 0$,*

$$q_{\alpha, \mathfrak{m}}^{(k)}(\mathcal{A}'\mathfrak{m}^{s_\alpha}) = 0.$$

Proof. Denote by $\mathfrak{m}(\alpha) \subset \mathfrak{m}$ the parabolic subalgebra of \mathfrak{m} , generated as a vector space by all root vectors of \mathfrak{m} except e_α . Due to (4.3), (i), it is sufficient to prove that $q_{\alpha, k}(\mathcal{A}\mathfrak{m}(\alpha)) = 0$. The basic theory of root systems for simple Lie algebras says, that $\hat{e}_{\pm\alpha}(\mathfrak{m}(\alpha)) \subset \mathfrak{m}(\alpha)$. Thus for any $n \geq 0$ we have $\hat{e}^n(\mathcal{A}\mathfrak{m}(\alpha)) \subset \mathcal{A}\mathfrak{m}(\alpha)$ and $\mathfrak{m}(\alpha)e_\alpha^n g_{n, \alpha}^{-1} \subset \mathcal{A}'\mathfrak{m}(\alpha)$, which imply the statement of the Lemma. \square

Due to Lemma 4.3, the maps $q_{\alpha, \mathfrak{m}}^{(k)}$ induce linear maps of $\mathcal{A}'/\mathcal{A}'\mathfrak{m}^{s_\alpha}$ to $\mathcal{A}'/\mathcal{A}'\mathfrak{m}$. We denote them by the same symbol:

$$q_{\alpha, \mathfrak{m}}^{(k)} : \mathcal{A}'/\mathcal{A}'\mathfrak{m}^{s_\alpha} \rightarrow \mathcal{A}'/\mathcal{A}'\mathfrak{m}.$$

Proposition 4.4.

- (i) *The map $q_{\alpha, \mathfrak{m}}$ transforms the normalizer $Nr(\mathcal{A}'\mathfrak{m}^{s_\alpha})$ to the Mickelsson algebra $Z^m(\mathcal{A}) = Nr(\mathcal{A}'\mathfrak{m})/\mathcal{A}'\mathfrak{m}$.*
- (ii) *The restriction of the map $q_{\alpha, \mathfrak{m}}$ to the normalizer $Nr(\mathcal{A}'\mathfrak{m}^{s_\alpha})$ defines an isomorphism of vector spaces $Z^{m^{s_\alpha}}(\mathcal{A})$ and $Z^m(\mathcal{A})$, satisfying (4.11) with $k = 0$.*

Proof. We prove first the statement (i). Let $\Delta_+(\mathfrak{m})$ be a system of positive roots, related to the decomposition $\mathfrak{g} = \mathfrak{m}_- + \mathfrak{h} + \mathfrak{m}$. By assumption, α is a simple root of $\Delta_+(\mathfrak{m})$.

Let x be an element of the normalizer of the ideal $\mathcal{A}'\mathfrak{m}^{s_\alpha}$, $x \in Nr(\mathcal{A}'\mathfrak{m}^{s_\alpha})$. It means that $e_{-\alpha}x \in \mathcal{A}'\mathfrak{m}^{s_\alpha}$ and $e_\gamma x \in \mathcal{A}'\mathfrak{m}^{s_\alpha}$ for any root $\gamma \in \Delta_+(\mathfrak{m})$, $\gamma \neq \alpha$.

Since $e_\alpha q_{\alpha, \mathfrak{m}} = 0$ by (4.3), (iv), we have to prove that

$$(4.12) \quad e_\gamma q_{\alpha, \mathfrak{m}}(x) = 0 \quad \text{for any } \gamma \in \Delta_+(\mathfrak{m}) \setminus \alpha.$$

Fix a positive root $\gamma \in \Delta_+(\mathfrak{m}) \setminus \alpha$. Let $\gamma_0, \dots, \gamma_m$ be an ' α -string' of roots, starting with γ , that is $\gamma_0 = \gamma$, and $\gamma_{k+1} = \gamma_k + \alpha$. Since α is simple, all roots γ_k are positive, $\gamma_k \in \Delta_+(\mathfrak{m}) \setminus \alpha$ and for any $k = 0, \dots, m$ we have

$$(4.13) \quad \hat{e}_\alpha^k(e_\gamma) = a_k e_{\gamma_k}, \quad k = 0, \dots, m, \quad a_k \in \mathbb{C}, \quad \hat{e}_\alpha^k(e_\gamma) = 0, \quad k > m.$$

For any $y \in \mathcal{A}$ we have by (4.13)

$$q_\alpha(e_\gamma y) = \sum_{k=0}^m a_k \sum_{n=k}^{\infty} \frac{(-1)^n}{(n-k)!} e_{\gamma_k} \hat{e}_\alpha^n(y) e_{-\alpha}^n \cdot g_{n,\alpha},$$

therefore,

$$(4.14) \quad e_\gamma q_\alpha(y) = q_\alpha(e_\gamma y) - \sum_{k=1}^m a_k e_{\gamma_k} q_\alpha^{(k)}(y).$$

Iterations of (4.14) and a factorization by $\mathcal{A}'\mathfrak{m}$ give the relation

$$(4.15) \quad e_\gamma q_{\alpha,\mathfrak{m}}(x) = q_{\alpha,\mathfrak{m}}(e_\gamma x) + \sum_{k=1}^m b_k q_{\alpha,\mathfrak{m}}^{(k)}(e_{\gamma_k} x), \quad b_k \in \mathbb{C}.$$

The right hand side of (4.15) is zero by assumption. This proves (4.12) and the statement (i) of the Proposition.

Let us prove (ii). The root $-\alpha$ is a simple positive root for the algebra \mathfrak{m}^{s_α} . Thus by (i) the map $q_{-\alpha,\mathfrak{m}^{s_\alpha}}$ maps the normalizer $\text{Nr}(\mathcal{A}'\mathfrak{m})$ and the Mickelsson algebra $Z^{\mathfrak{m}}(\mathcal{A})$ to the Mickelsson algebra $Z^{\mathfrak{m}^{s_\alpha}}(\mathcal{A})$. This implies that the map

$$q'_{-\alpha,\mathfrak{m}^{s_\alpha}} = \text{Ad}_{h_\alpha+1}^{-1} \cdot q_{-\alpha,\mathfrak{m}^{s_\alpha}},$$

which sends $x \in \text{Nr}(\mathcal{A}'\mathfrak{m})$ to $(h_\alpha + 1)^{-1} \cdot q_{-\alpha,\mathfrak{m}^{s_\alpha}}(x) \cdot (h_\alpha + 1)$, also maps $Z^{\mathfrak{m}}(\mathcal{A})$ to the Mickelsson algebra $Z^{\mathfrak{m}^{s_\alpha}}(\mathcal{A})$. Proposition 4.2 says that $q'_{-\alpha,\mathfrak{m}^{s_\alpha}}$ is inverse to the map of $Z^{\mathfrak{m}^{s_\alpha}}(\mathcal{A})$ to $Z^{\mathfrak{m}}(\mathcal{A})$, induced by the restriction of $q_{\alpha,\mathfrak{m}}$ to $\text{Nr}(\mathcal{A}'\mathfrak{m}^{s_\alpha})$. This proves the statement (ii). \square

4.3. Maps $q_{\overline{w},\mathfrak{m}}$. Let \mathfrak{m} be a maximal nilpotent subalgebra of \mathfrak{g} , conjugated to \mathfrak{n} by an automorphism $T_{w'}$, where $w' \in W$, $\mathfrak{m} = T_{w'}(\mathfrak{n})$, and $\mathfrak{m}_- = T_{w'}(\mathfrak{n}_-)$ the opposite maximal nilpotent subalgebra. Assume that $w \in W$ satisfies the condition

$$(4.16) \quad \dim T_w(\mathfrak{m}) \cap \mathfrak{m}_- = l(w),$$

where $l(w)$ is the length of w in W . Since $\mathfrak{m} = T_{w'}(\mathfrak{n})$, the condition (4.16) is equivalent to the relation $l(w'w) = l(w') + l(w)$. Let \overline{w} be a reduced decomposition of the element $w \in W$:

$$(4.17) \quad \overline{w} = \{w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_n}}\}.$$

Let $n = l(w)$. Define a sequence $w_1, w_2, \dots, w_n, w_n = w$ of elements of the Weyl group W , a sequence $\gamma_1, \gamma_2, \dots, \gamma_n$ of positive roots, and a sequence $\mathfrak{m}_0, \dots, \mathfrak{m}_n$ of maximal nilpotent subalgebras of \mathfrak{g} by the following inductive rule :

$$(4.18) \quad w_0 = 1, \quad w_{k+1} = w_k \cdot s_{\alpha_{i_{k+1}}},$$

$$(4.19) \quad \gamma_k = w' w_{k-1}(\alpha_{i_k}), \quad \mathfrak{m}_k = T_{w' w_{k-1}}(\mathfrak{n}).$$

The relations (4.16), (4.18)–(4.19) imply that for any $k = 1, \dots, n$ the root vector e_{γ_k} is a simple positive root vector for the algebra \mathfrak{m}_k , and the composition

$$(4.20) \quad q_{\overline{w},\mathfrak{m}} = q_{\gamma_1,\mathfrak{m}_1} \cdot q_{\gamma_2,\mathfrak{m}_2} \cdots q_{\gamma_n,\mathfrak{m}_n}$$

is a well defined map $q_{\bar{w}, \mathbf{m}} : \mathcal{A}' / \mathcal{A}' \mathbf{m}^w \rightarrow \mathcal{A}' / \mathcal{A}' \mathbf{m}$. The index \bar{w} reminds that this map is related by the construction to a reduced decomposition (4.17).

Denote by $\Delta_w(\mathbf{m})$ the set of roots $\gamma_1, \dots, \gamma_n$, defined in (4.19). Alternatively, $\Delta_w(\mathbf{m})$ consists of all roots $\gamma \in \Delta(\mathbf{m}) = w'(\Delta_+)$, such that $w^{-1}(\gamma) \in \Delta(\mathbf{m}_-) = w'(\Delta_-)$.

Lemma 4.5. (See [Zh], Section 5.2.4) *Let $x \in \mathcal{A}'$. Then, in the notation (4.18)-(4.20), we have*

$$\begin{aligned} (i) \quad & q_{\bar{w}, \mathbf{m}}(xd) = q_{\bar{w}, \mathbf{m}}(x) \cdot \tau_{w'(\rho) - w'w(\rho)}(d) && \text{for any } d \in D, \\ (ii) \quad & e_\alpha \cdot q_{\bar{w}, \mathbf{m}}(x) = 0 && \text{for any } \alpha \in \Delta_w(\mathbf{m}), \\ (iii) \quad & q_{\bar{w}, \mathbf{m}}(e_\alpha x) = 0 && \text{for any } \alpha \in \Delta_w(\mathbf{m}), \\ (iv) \quad & e_{w'w(\alpha)} q_{\bar{w}, \mathbf{m}}(x) = q_{\bar{w}, \mathbf{m}}(e_{w'w(\alpha)} x) && \text{if } l(w'ws_\alpha) > l(w'w), \\ (v) \quad & e_{w'(\alpha)} q_{\bar{w}, \mathbf{m}^{s_\alpha}}(x) = q_{\bar{w}, \mathbf{m}^{s_\alpha}}(e_{w'(\alpha)} x) && \text{if } l(w's_\alpha w) > l(w'w). \end{aligned}$$

Proof. The property (i) is a direct consequence of (4.11), (ii). Indeed, the relation (4.11), (ii), for $k = 0$ implies that for any $x \in \mathcal{A}'$ and $d \in D$ we have

$$q_{\bar{w}, \mathbf{m}}(xd) = q_{\bar{w}, \mathbf{m}}(x) \tau_{\gamma_1 + \dots + \gamma_n}(d) = q_{\bar{w}, \mathbf{m}}(x) \cdot \tau_{w'(\rho) - w'w(\rho)}(d).$$

Suppose that the relations (ii)-(iv) take place for all $\mathbf{m} = w'(\mathbf{n})$, and all reduced decompositions of any element w of the Weyl group of the length less than n , such that $l(w'w) = l(w') + l(w)$. Let (4.17) be a reduced decomposition of the element w of the length n .

In the notation of (4.18)-(4.20) we have the equality $q_{\bar{w}, \mathbf{m}} = q_{\bar{w}_{n-1}, \mathbf{m}} q_{\gamma_n, \mathbf{m}_n}$, with $l(w_{n-1}) = n - 1$. Thus, by the induction assumption, for any $x \in \mathcal{A}'$, $e_{\gamma_i} q_{\bar{w}_{n-1}, \mathbf{m}}(x) = 0$ for $i = 1, \dots, n - 1$ and $e_{\gamma_n} q_{\bar{w}_{n-1}, \mathbf{m}}(x) = q_{\bar{w}_{n-1}, \mathbf{m}} e_{\gamma_n}(x)$. This implies equalities $e_{\gamma_i} q_{\bar{w}, \mathbf{m}}(x) = 0$ for $i = 1, \dots, n - 1$ and

$$e_{\gamma_n} q_{\bar{w}, \mathbf{m}}(x) = q_{\bar{w}_{n-1}, \mathbf{m}}(x) e_{\gamma_n} q_{\gamma_n, \mathbf{m}_n}(x).$$

The last line is zero due to (4.11), (iii). This proves the induction step for the statement (ii).

On the other hand, present w as a product $w = s_{\gamma_1} w''$, where w'' is an element of the length $n - 1$ with a given reduced decomposition $\bar{w}'' = \{w'' = s_{\alpha_{i_2}} \cdot \dots \cdot s_{\alpha_{i_n}}\}$. We have a decomposition $q_{\bar{w}, \mathbf{m}} = q_{\gamma_1, \mathbf{m}} \cdot q_{\bar{w}'', \mathbf{m}_2}$, where

$$(4.21) \quad q_{\bar{w}'', \mathbf{m}_2} = q_{\gamma_2, \mathbf{m}_2} q_{\gamma_3, \mathbf{m}_3} \cdot \dots \cdot q_{\gamma_n, \mathbf{m}_n}.$$

The induction assumptions say that $q_{\bar{w}'', \mathbf{m}_2}(e_{\gamma_i} x) = 0$ for $i = 2, \dots, n$ by (ii) and

$$q_{\bar{w}'', \mathbf{m}_2}(e_{\gamma_1} x) = e_{\gamma_1} q_{\bar{w}'', \mathbf{m}_2}(x)$$

by (v). Thus $q_{\bar{w}, \mathbf{m}}(e_{\gamma_i} x) = 0$ for $i = 2, \dots, n$ and

$$q_{\bar{w}, \mathbf{m}}(e_{\gamma_1} x) = q_{\gamma_1, \mathbf{m}} q_{\bar{w}'', \mathbf{m}_2}(e_{\gamma_1} x) = q_{\gamma_1, \mathbf{m}}(e_{\gamma_1} q_{\bar{w}'', \mathbf{m}_2}(x)) = 0$$

by (4.3), (iv). This proves the induction step for (iii).

Let us prove the induction step for (iv). Take a simple root α , such that $l(w'ws_\alpha) > l(w'w)$. Let $\gamma = w'w(\alpha)$. Then γ is a positive root and the sequence

$$(4.22) \quad \gamma_1, \dots, \gamma_n, \gamma$$

is convex, that is, the sum of any two elements of the sequence lies between them, if the sum is a root. Let μ_0, \dots, μ_m be the finite " γ_1 -sequence' of positive roots, starting with γ , that is, $\mu_0 = \gamma$, $\mu_{k+1} = \mu_k + \gamma_1$. Then each μ_k belongs to the set (4.22),

$\mu_k = \gamma_{i_k}$, where $i_k \in \{2, \dots, n\}$ if $k > 0$, and $\hat{e}_\alpha^k(e_\gamma) = a_k e_{\mu_k}$ with $a_k \in \mathbb{C}$, and $\hat{e}_\alpha^k(e_\gamma) = 0$ for $k > m$. This implies, see the proof of Proposition (4.4), that

$$(4.23) \quad e_\gamma q_{\gamma_1}(y) = q_{\gamma_1}(e_\gamma y) + \sum_{k=1}^m b_k q_{\gamma_1}^{(k)}(e_{\mu_k} y), \quad b_k \in \mathbb{C}.$$

The relation (4.23) implies the equality

$$e_\gamma q_{\overline{w}, m}(x) = q_{\gamma_1, m}(e_\gamma q_{\overline{w}'', m_2}(x)) + \sum_{k=1}^m b_k q_{\gamma_1, m}^{(k)}(e_{\mu_k} q_{\overline{w}'', m_2}(x)),$$

where $q_{\overline{w}'', m_2}$ is defined in (4.21). The induction assumption says that $e_\gamma q_{\overline{w}'', m_2}(x) = q_{\overline{w}'', m_2}(e_\gamma x)$ and $e_{\mu_k} q_{\overline{w}'', m_2} = 0$ for any $k \geq 1$. This implies the induction step for (iv). The statement (v) is proved in an analogous manner.

4.4. Map q_{w_0} . In this Section we assume that \mathfrak{g} is a finite-dimensional reductive Lie algebra and \mathcal{A} is a \mathfrak{g} -admissible algebra. In this case the Weyl group W is finite and the adjoint action of \mathfrak{g} in \mathcal{A} is locally finite.

Set $\mathbf{m} = \mathbf{n}$ and for any $w \in W$ and a reduced decomposition $\overline{w} = \{w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_k}}\}$ denote the map $q_{\overline{w}, \mathbf{n}}$ as $q_{\overline{w}}$:

$$q_{\overline{w}} \equiv q_{\overline{w}, \mathbf{n}}.$$

Let $\overline{w}_0 = \{w_0 = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_n}}\}$ be a reduced decomposition of the longest element $w_0 \in W$. The map $q_{\overline{w}_0}$ has the following properties: the right ideal $\mathbf{n}\mathcal{A}'$ is in the kernel of $q_{\overline{w}_0}$ by Lemma 4.5, (ii), and the image of $q_{\overline{w}_0}$ is in the normalizer of the left ideal $\mathcal{A}'\mathbf{n}$ by Lemma 4.5, (iii). It means that it induces the map of the double coset space ${}_{\mathbf{n}}\mathcal{A}_{\mathbf{n}-} = \mathbf{n}\mathcal{A}' \backslash \mathcal{A}' / \mathcal{A}'\mathbf{n}_{-}$ to the Mickelsson algebra $Z^n(\mathcal{A})$:

$$q_{\overline{w}_0} : {}_{\mathbf{n}}\mathcal{A}_{\mathbf{n}-} \rightarrow Z^n(\mathcal{A}).$$

Let $\mathcal{V} \subset \mathcal{A}$ be an ad -invariant generating subspace of \mathcal{A} . Let z'_w , where $w \in \mathcal{V}$, be the canonical generators of the D -module $Z^n(\mathcal{A})$, see Theorem 1.

Proposition 4.6. *The mapping $q_{\overline{w}_0}$ of vector spaces defines an isomorphism of the double coset space ${}_{\mathbf{n}}\mathcal{A}_{\mathbf{n}-}$ and $Z^n(\mathcal{A})$, such that for any $x \in {}_{\mathbf{n}}\mathcal{A}_{\mathbf{n}-}$, $d \in D$ and $v \in \mathcal{V}$ we have*

$$(4.24) \quad [d, q_{\overline{w}_0}(x)] = q_{\overline{w}_0}([d, x]), \quad q_{\overline{w}_0}(xd) = q_{\overline{w}_0}(x)\tau_{\rho-w_0(\rho)}(d),$$

$$(4.25) \quad q_{\overline{w}_0}(v) = z'_v.$$

Proof. The arguments are the same as in the proof of Proposition 4.1. The double coset space ${}_{\mathbf{n}}\mathcal{A}_{\mathbf{n}-}$ is a free D -module, generated by the vector space \mathcal{V} . On the other hand, the Mickelsson algebra $Z^n(\mathcal{A})$ is also a free D -module, generated by the vectors z'_v , $v \in \mathcal{V}$, see Theorem 1. By the structure of the map $q_{\overline{w}_0}$,

$$q_{\overline{w}_0}(v) = w + \sum v_i g_i,$$

where $g_i \in U'(\mathfrak{g})$, and $v_i \in \mathcal{V}$ have the weight strictly bigger than the weight of v , that is, $\mu(v - v_i) \in Q_+$, $\mu(v - v_i) \neq 0$, where $\mu(x) \in \mathfrak{h}^*$ denotes the weight of x . We can present further any g_i as a sum $g_i = \sum_j f_{i,j} d_{i,j} e_{i,j}$, where $f_{i,j} \in U(\mathfrak{n}_{-})$, $d_{i,j} \in D$ and $e_{i,j} \in U(\mathfrak{n})$. The terms, where $e_{i,j} \neq 1$, vanish by definition in $Z^n(\mathcal{A})$, so we have the equality (4.25). \square

Proposition 4.6 implies that the map $q_{\overline{w}_0}$ does not depend on a reduced decomposition of w_0 . We denote it further by q_{w_0} ,

$$q_{w_0} \equiv q_{\overline{w}_0}.$$

Corollary 4.7. *The restriction of the map q_{w_0} to the normalizer $Nr(\mathcal{A}'\mathfrak{n}_-)$ defines an isomorphism of vector spaces $Z^{\mathfrak{n}_-}(\mathcal{A})$ and $Z^{\mathfrak{n}}(\mathcal{A})$, such that*

$$(4.26) \quad \begin{aligned} [d, q_{\overline{w}_0}(x)] &= q_{\overline{w}_0}([d, x]), & q_{w_0}(xd) &= q_{w_0}(x)\tau_{\rho-w_0(\rho)}(d), & d &\in D, \\ q_{w_0}(z_{\mathfrak{n}_-,v}) &= z'_{\mathfrak{n},v}, & & & v &\in \mathcal{V}. \end{aligned}$$

Here $z_{\mathfrak{n}_-,v}$ are generators of the Mickelsson algebra $Z^{\mathfrak{n}_-}(\mathcal{A})$ of the 'first type', see Proposition 3.3, $z'_{\mathfrak{n},v} \equiv z'_v$ are generators of the Mickelsson algebra $Z^{\mathfrak{n}}(\mathcal{A})$ of the 'second type', see Proposition 3.4.

Proof. We have $\mathfrak{n}_- = \mathfrak{n}^{w_0}$ and all the statements of the Corollary follow by induction from Proposition 4.4. We should prove only the equality (4.26). By the definition (3.9), the element $z_{\mathfrak{n}_-,v} \in Z^{\mathfrak{n}_-}(\mathcal{A})$ has a form

$$z_{\mathfrak{n}_-,v} = v + \sum_{i=1} d_i e_i v_i, \quad e_i \in \mathfrak{n}U(\mathfrak{n}), \quad d_i \in D, \quad v_i \in \mathcal{V},$$

that is, $z_{\mathfrak{n}_-,v} = v \pmod{\mathfrak{n}\mathcal{A}'}$. By the properties of the map q_{w_0} , we have $q_{w_0}(z_{\mathfrak{n}_-,v}) = q_{w_0}(v) = z'_v$. \square

4.5. Cocycle properties. In this Section \mathfrak{g} is an arbitrary contragredient Lie algebra of finite growth, \mathcal{A} is a g -admissible algebra, \mathfrak{m} a maximal nilpotent subalgebra of \mathfrak{g} , conjugated to \mathfrak{n} by an element of the Weyl group W , and w an element of W , satisfying the condition (4.16).

Proposition 4.8. *The maps $q_{\overline{w},\mathfrak{m}}$ do not depend on a choice of the reduced decompositions of $w \in W$.*

Proof. First we take an element $w \in W$, equal to the longest element of a reductive subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of rank two. Then the algebra $\mathfrak{g}' = \mathfrak{n}'_- + \mathfrak{h} + \mathfrak{n}'$ is generated by the elements $e_{\pm\gamma_i}$ and $h \in \mathfrak{h}$, where $\gamma_1, \dots, \gamma_n$ are the members of the sequence (4.19), such that all $e_{\gamma_i} \in \mathfrak{m}$ by the condition (4.16). By Proposition 4.6, the map

$$q_{\overline{w},\mathfrak{n}'} : \mathcal{A}' \rightarrow \mathcal{A}'/\mathcal{A}'\mathfrak{n}'$$

does not depend on a reduced decomposition of w . The map

$$q_{\overline{w},\mathfrak{m}} : \mathcal{A}' \rightarrow \mathcal{A}'/\mathcal{A}'\mathfrak{m}$$

is the composition of $q_{\overline{w},\mathfrak{n}'}$ and the natural projection of $\mathcal{A}'/\mathcal{A}'\mathfrak{n}'$ to $\mathcal{A}'/\mathcal{A}'\mathfrak{m}$ and thus also does not depend on a choice of the reduced decomposition of w .

This result implies the statement of the proposition for general w , since any two reduced decompositions are related by a sequence of flips of reduced decompositions of longest elements of rank two subalgebras. \square

With the use of Proposition 4.8, we simplify further the notation for the maps $q_{\overline{w},\mathfrak{m}}$ and $q_{\overline{w}}$ and write them as $q_{w,\mathfrak{m}}$ and q_w :

$$q_{w,\mathfrak{m}} \equiv q_{\overline{w},\mathfrak{m}}, \quad q_w \equiv q_{\overline{w}} \equiv q_{\overline{w},\mathfrak{n}}.$$

Proposition 4.8 can be formulated as the condition

$$(4.27) \quad q_{w'w,\mathfrak{m}} = q_{w',\mathfrak{m}}q_{w,\mathfrak{m}^{w'}}, \quad \text{if } l(w'w) = l(w') + l(w).$$

This statement is known as 'Zhelobenko cocycle condition'.

5. HOMOMORPHISM PROPERTIES OF ZHELOBENKO MAPS

5.1. Homomorphism property of maps q_α . Let \mathcal{A} be an admissible algebra over a contragredient Lie algebra \mathfrak{g} of finite growth. Let α be a real root of \mathfrak{g} , \mathfrak{g}_α the \mathfrak{sl}_2 -subalgebra of \mathfrak{g} , generated by $e_{\pm\alpha}$ and h_α , $\mathfrak{n}_{\pm\alpha} = \mathbb{C}e_{\pm\alpha}$. Since the algebra \mathcal{A} is \mathfrak{g} -admissible and the adjoint action of \mathfrak{g}_α in \mathfrak{g} is locally finite, \mathcal{A} is \mathfrak{g}_α -admissible as well, see Section 4.1. In this setting we proved in Section 3.3 that the double coset space ${}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}} = \mathfrak{n}_\alpha\mathcal{A}'\backslash\mathcal{A}'/\mathcal{A}'\mathfrak{n}_{-\alpha}$ can be equipped with a structure of an associative algebra with the multiplication rule

$$a \circ b = ap_{-\alpha}(b) = \bar{p}_{-\alpha}(a)b,$$

in the notation of Section 2.2 and Section 3.3.

The following theorem is the basic point for applications of the Zhelobenko operators in the representation theory of Mickelsson algebras.

Theorem 3. *The map $q_\alpha : {}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}} \rightarrow Z^{\mathfrak{n}_\alpha}(\mathcal{A})$ is a homomorphism of algebras.*

Theorem 3 and Proposition 4.1 imply that q_α establishes an *isomorphism* of the double coset algebra ${}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}}$ and Mickelsson algebra $Z^{\mathfrak{n}_\alpha}(\mathcal{A})$. On the other hand, Theorem 3 implies the equality

$$(5.1) \quad q_\alpha(xy) = q_\alpha(x)q_\alpha(y), \quad \text{for any } x \in \mathcal{A}', y \in \text{Nr}(\mathcal{A}'\mathfrak{n}_{-\alpha}).$$

Indeed, if $y \in \text{Nr}(\mathcal{A}'\mathfrak{n}_{-\alpha})$ and \bar{y} is the class of y in $\mathcal{A}'/\mathcal{A}'\mathfrak{n}_{-\alpha}$ then $\bar{y} = p_{-\alpha}(\bar{y})$ and the first equality in (4.12) is a corollary of the first statement of Theorem 3.

Proof. Let $P_{-\alpha}$ be the extremal projector, related to the decomposition $\mathfrak{g}_\alpha = \mathfrak{n}_\alpha + \mathbb{C}h_\alpha + \mathfrak{n}_{-\alpha}$ of the algebra \mathfrak{g}_α . It is given by the relations (2.12) and (2.13). The corresponding operators $p_{-\alpha} : \mathcal{A}'/\mathfrak{n}_{-\alpha}\mathcal{A}' \rightarrow \mathcal{A}'/\mathfrak{n}_{-\alpha}\mathcal{A}'$ and $\bar{p}_{-\alpha} : \mathcal{A}'\mathfrak{n}_\alpha\backslash\mathcal{A}' \rightarrow \mathcal{A}'\mathfrak{n}_\alpha\backslash\mathcal{A}'$ can be written as

$$(5.2) \quad p_{-\alpha}(x) = \sum_{m \geq 0} \frac{1}{m!} \frac{1}{(h_\alpha - 2) \cdots (h_\alpha - m - 1)} e_\alpha \hat{e}_{-\alpha}^m(x) \quad \text{mod } \mathcal{A}'e_{-\alpha},$$

$$(5.3) \quad \bar{p}_{-\alpha}(x) = \sum_{m \geq 0} \frac{(-1)^m}{m!} \hat{e}_\alpha^m(x) e_{-\alpha}^m \frac{1}{(h_\alpha - 2) \cdots (h_\alpha - m - 1)} \quad \text{mod } e_\alpha \mathcal{A}'.$$

We should establish an equality

$$(5.4) \quad q_\alpha(\bar{p}_{-\alpha}(x)y) = q_\alpha(x)q_\alpha(y) \quad \text{for any } x \in \mathfrak{n}_\alpha\mathcal{A}'\backslash\mathcal{A}', \quad y \in \mathcal{A}'.$$

Suppose $y \in \mathcal{A}$ is a weight vector of the weight $\mu_y \in \mathfrak{h}^*$, such that $[h_\alpha, y] = \mu_y y$ for any $h \in \mathfrak{h}$. We have

$$\begin{aligned}
(5.5) \quad q_\alpha(\bar{p}_{-\alpha}(x)y) &= q_\alpha\left(\sum_{m \geq 0} \frac{(-1)^m}{m!} \hat{e}_\alpha^m(x) e_{-\alpha}^m \frac{1}{(h_\alpha-2) \cdots (h_\alpha-m-1)} y\right) \\
&= q_\alpha\left(\sum_{m \geq 0} \frac{(-1)^m}{m!} \hat{e}_\alpha^m(x) e_{-\alpha}^m y \frac{1}{(h_\alpha-2+\mu_y) \cdots (h_\alpha-m-1+\mu_y)}\right) \\
&= q_\alpha\left(\sum_{m \geq 0} \frac{(-1)^m}{m!} \hat{e}_\alpha^m(x) e_{-\alpha}^m y\right) \frac{1}{(h_\alpha+\mu_y) \cdots (h_\alpha-m+1+\mu_y)} \\
&= \sum_{n, m \geq 0} \frac{(-1)^{n+m}}{n!m!} \hat{e}_\alpha^n(\hat{e}_\alpha^m(x) e_{-\alpha}^m y) e_{-\alpha}^n \frac{1}{g_{n,\alpha}} \frac{1}{(h_\alpha+\mu_y) \cdots (h_\alpha-m+1+\mu_y)} \\
&= \sum_{n, m \geq 0} \frac{(-1)^{n+m}}{n!m!} \\
&\quad \cdot \sum_{k=0}^n \binom{n}{k} \hat{e}_\alpha^k(\hat{e}_\alpha^m(x) e_{-\alpha}^m) \hat{e}_\alpha^{n-k}(y) e_{-\alpha}^n \frac{1}{g_{n,\alpha}} \frac{1}{(h_\alpha+\mu_y) \cdots (h_\alpha-m+1+\mu_y)} \\
&= \sum_{k, m=0}^\infty \frac{(-1)^m}{m!k!} \hat{e}_\alpha^k(\hat{e}_\alpha^m(x) e_{-\alpha}^m) \frac{1}{(h_\alpha+2k) \cdots (h_\alpha+2k-m+1)} q_\alpha^{(k)}(y).
\end{aligned}$$

The sum of the terms with $k = 0$ in (5.5) is $q_\alpha(x) \cdot q_\alpha(y)$. So it is sufficient to prove the following identity in $\mathcal{A}'/\mathcal{A}'e_\alpha$:

$$(5.6) \quad \sum_{k=1}^\infty \sum_{m=0}^\infty \frac{(-1)^m}{m!k!} \hat{e}_\alpha^k(\hat{e}_\alpha^m(x) e_{-\alpha}^m) \frac{1}{(h_\alpha+2k) \cdots (h_\alpha+2k-m+1)} q_\alpha^{(k)}(y) = 0$$

for any $x, y \in \mathcal{A}$.

We have for any $k \geq 1$:

$$\begin{aligned}
(5.7) \quad &\hat{e}_\alpha^k(\hat{e}_\alpha^m(x) e_{-\alpha}^m) \\
&= \hat{e}_\alpha^{k-1}(\hat{e}_\alpha^{m+1}(x) e_{-\alpha}^m + m \hat{e}_\alpha^m(x) e_{-\alpha}^{m-1} (h_\alpha - m + 1)) \\
&= \hat{e}_\alpha^{k-1}(\hat{e}_\alpha^{m+1}(x) e_{-\alpha}^m) + m \hat{e}_\alpha^{k-1}(\hat{e}_\alpha^m(x) e_{-\alpha}^{m-1}) (h_\alpha - m + 1) \\
&\quad - 2m(k-1) \hat{e}_\alpha^{k-2}(\hat{e}_\alpha^m(x) e_{-\alpha}^{m-1}) e_\alpha.
\end{aligned}$$

Denote the left hand side of (5.6) by S . Substitute (5.7) into S . We get

$$\begin{aligned}
S &= \sum_{k=1}^\infty \sum_{m=0}^\infty \frac{(-1)^m}{m!k!} \\
&\quad \cdot \left(\hat{e}_\alpha^{k-1}(\hat{e}_\alpha^{m+1}(x) e_{-\alpha}^m) \frac{1}{(h_\alpha+2k) \cdots (h_\alpha+2k-m+1)} q_\alpha^{(k)}(y) \right. \\
&\quad + m \hat{e}_\alpha^{k-1}(\hat{e}_\alpha^m(x) e_{-\alpha}^{m-1}) \frac{(h_\alpha - m + 1)}{(h_\alpha+2k) \cdots (h_\alpha+2k-m+1)} q_\alpha^{(k)}(y) \\
&\quad \left. - 2m(k-1) \hat{e}_\alpha^{k-2}(\hat{e}_\alpha^m(x) e_{-\alpha}^{m-1}) e_\alpha \frac{1}{(h_\alpha+2k) \cdots (h_\alpha+2k-m+1)} q_\alpha^{(k)}(y) \right).
\end{aligned}$$

Substitute $(h_\alpha - m + 1) = (h_\alpha + 2k - m + 1) - 2k$ into the second sum and use the relation (4.3), (iv) in the third sum. We get

$$\begin{aligned}
S = & \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!k!} \\
& \cdot \left(\hat{e}_\alpha^{k-1} (\hat{e}_\alpha^{m+1}(x) e_{-\alpha}^m) \frac{1}{(h_\alpha + 2k) \cdots (h_\alpha + 2k - m + 1)} q_\alpha^{(k)}(y) \right. \\
& + m \hat{e}_\alpha^{k-1} (\hat{e}_\alpha^m(x) e_{-\alpha}^{m-1}) \frac{1}{(h_\alpha + 2k) \cdots (h_\alpha + 2k - m + 2)} q_\alpha^{(k)}(y) \\
& - 2km \hat{e}_\alpha^{k-1} (\hat{e}_\alpha^m(x) e_{-\alpha}^{m-1}) \frac{1}{(h_\alpha + 2k) \cdots (h_\alpha + 2k - m + 1)} q_\alpha^{(k)}(y) \\
& \left. + 2mk(k-1) \hat{e}_\alpha^{k-2} (\hat{e}_\alpha^m(x) e_{-\alpha}^{m-1}) \frac{1}{(h_\alpha + 2k - 2) \cdots (h_\alpha + 2k - m - 1)} q_\alpha^{(k-1)}(y) \right).
\end{aligned}$$

Change indices of summation: m to $m+1$ in the second and third sums; m to $m+1$ and k to $k+1$ in the last sum. Then

$$\begin{aligned}
S = & \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!k!} \\
& \cdot \left(\hat{e}_\alpha^{k-1} (\hat{e}_\alpha^{m+1}(x) e_{-\alpha}^m) \frac{1}{(h_\alpha + 2k) \cdots (h_\alpha + 2k - m + 1)} q_\alpha^{(k)}(y) \right. \\
& - \hat{e}_\alpha^{k-1} (\hat{e}_\alpha^{m+1}(x) e_{-\alpha}^m) \frac{1}{(h_\alpha + 2k) \cdots (h_\alpha + 2k - m + 1)} q_\alpha^{(k)}(y) \\
& + 2k \hat{e}_\alpha^{k-1} (\hat{e}_\alpha^m(x) e_{-\alpha}^{m-1}) \frac{1}{(h_\alpha + 2k) \cdots (h_\alpha + 2k - m)} q_\alpha^{(k)}(y) \\
& \left. - 2k \hat{e}_\alpha^{k-2} (\hat{e}_\alpha^m(x) e_{-\alpha}^{m-1}) \frac{1}{(h_\alpha + 2k) \cdots (h_\alpha + 2k - m)} q_\alpha^{(k)}(y) \right) = 0.
\end{aligned}$$

The theorem is proved. \square

5.2. Properties of maps $q_{\alpha, \mathbf{m}}$ and $q_{w, \mathbf{m}}$. Let α be a real root of the Lie algebra \mathfrak{g} , e_α the corresponding root vector with respect to the decomposition (2.1) and $\mathbf{m} = \mathbf{n}^w$, where $w \in W$, a maximal nilpotent subalgebra of \mathfrak{g} , such that e_α is a simple positive root vector of \mathbf{m} .

Theorem 4. *For any $x \in \mathcal{A}'$, $y \in \text{Nr}(\mathcal{A}' \mathbf{m}^{s_\alpha})$*

$$q_{\alpha, \mathbf{m}}(xy) = q_{\alpha, \mathbf{m}}(x) q_{\alpha, \mathbf{m}}(y).$$

Proof. Since $y \in \text{Nr}(\mathcal{A}' \mathbf{m}^{s_\alpha})$, its image \bar{y} in $M_{\mathbf{m}^{s_\alpha}}(\mathcal{A}') = \mathcal{A}' / \mathcal{A}' \mathbf{m}^{s_\alpha}$ satisfies the relations $e_\gamma \bar{y} = 0$ for all $\gamma \in \Delta_+(\mathbf{m}^{s_\alpha})$. In particular, we have the equality

$$(5.8) \quad e_{-\alpha} \bar{y} = 0 \quad \text{in } \mathcal{A}' / \mathcal{A}' \mathbf{m}^{s_\alpha}.$$

Let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be the \mathfrak{sl}_2 -subalgebra, generated by $e_{\pm\alpha}$ and h_α . Since \mathcal{A} is \mathfrak{g} -admissible, it is \mathfrak{g}_α -admissible as well and $M_{\mathbf{m}^{s_\alpha}}(\mathcal{A}')$ is locally nilpotent with respect to $e_{-\alpha}$. Thus the extremal projector $P_{-\alpha}$, related to $\mathbf{n}_{-\alpha} = \mathbb{C}e_{-\alpha}$, acts in $M_{\mathbf{m}^{s_\alpha}}(\mathcal{A}')$. Denote, following the notation of Section 2.2, its image in $\text{End } M_{\mathbf{m}^{s_\alpha}}(\mathcal{A}')$ by $p_{-\alpha}$. By (5.8) and the properties of the extremal projector, we have

$$(5.9) \quad \bar{y} = p_{-\alpha}(\bar{y}) \quad \text{in } \mathcal{A}' / \mathcal{A}' \mathbf{m}^{s_\alpha}.$$

The equality (5.9) can be read as

$$y \in \text{Nr}(\mathcal{A}'\mathbf{n}_{-\alpha}) \pmod{\mathcal{A}'\mathbf{m}^{s_\alpha}},$$

that is $y = y' + z$, where $y' \in \text{Nr}(\mathcal{A}'\mathbf{m}_{-\alpha})$ and $z \in \mathcal{A}'\mathbf{m}^{s_\alpha}$.

Indeed, $p_{-\alpha}(\bar{y}) \in \text{Nr}(\mathcal{A}'\mathbf{m}_{-\alpha}) \pmod{\mathcal{A}'\mathbf{m}_{-\alpha}}$ and $\mathcal{A}'\mathbf{m}_{-\alpha} \subset \mathcal{A}'\mathbf{m}^{s_\alpha}$. Due to Theorem 3, see (5.1), and the properties of the maps $q_{\alpha, \mathbf{m}}$, we have

$$q_{\alpha, \mathbf{m}}(xy) = q_{\alpha, \mathbf{m}}(xy') = q_{\alpha, \mathbf{m}}(x)q_{\alpha, \mathbf{m}}(y') = q_{\alpha, \mathbf{m}}(x)q_{\alpha, \mathbf{m}}(y).$$

□

Combining Theorem 3 with the statements of Proposition 4.4 and Proposition 7.3, we conclude that the restriction of the map $q_{\alpha, \mathbf{m}}$ to the normalizer $\text{Nr}(\mathcal{A}'\mathbf{m}^{s_\alpha})$ defines an isomorphism of the algebras $Z^{\mathbf{m}^{s_\alpha}}(\mathcal{A})$ and $Z^{\mathbf{m}}(\mathcal{A})$.

Iterations of these conclusions give the following statement.

Proposition 5.1. *For any $w', w \in W$, such that $l(w'w) = l(w') + l(w)$ and $\mathbf{m} = \mathbf{n}^{w'}$, the restriction of the map $q_{w, \mathbf{m}}$ to the normalizer $\text{Nr}(\mathcal{A}'\mathbf{m}^w)$ defines an isomorphism of the algebras $Z^{\mathbf{m}^w}(\mathcal{A})$ and $Z^{\mathbf{m}}(\mathcal{A})$ such that for any $x \in Z^{\mathbf{m}^w}(\mathcal{A})$ and $d \in D$ we have*

$$[d, q_{w, \mathbf{m}}(x)] = q_{w, \mathbf{m}}([d, x]), \quad q_{w, \mathbf{m}}(xd) = q_{w, \mathbf{m}}(x) \cdot \tau_{w'(\rho) - w'w(\rho)}(d).$$

The case of a finite-dimensional reductive Lie algebra \mathfrak{g} and the element w_0 of the maximal length is special. In this case the following statements hold:

Proposition 5.2.

- (i) *The map q_{w_0} defines an isomorphism of the double coset algebra ${}_n\mathcal{A}_{n-}$ and the Mickelsson algebra $Z^n(\mathcal{A})$ such that for any $d \in D$, $v \in \mathcal{V}$ and $x \in Z_{P_{n-}}(\mathcal{A})$ we have*

$$q_{w_0}(v) = z'_v, \quad [d, q_{w_0}(x)] = q_{w_0}([d, x]), \quad q_{w_0}(xd) = q_{w_0}(x)\tau_{\rho - w_0(\rho)}(d);$$

- (ii) *the restriction of the map q_{w_0} to the normalizer $\text{Nr}(\mathcal{A}'\mathbf{n}_{-})$ defines an isomorphism of the Mickelsson algebras $Z^{n-}(\mathcal{A})$ and $Z^n(\mathcal{A})$ such that for any $d \in D$, $v \in \mathcal{V}$ and $x \in Z^{n-}(\mathcal{A})$ we have*

$$q_{w_0}(z_{n-, v}) = z'_v, \quad [d, q_{w_0}(x)] = q_{w_0}([d, x]), \quad q_{w_0}(xd) = q_{w_0}(x)\tau_{\rho - w_0(\rho)}(d).$$

6. BRAID GROUP ACTION

6.1. Operators \check{q}_i . Suppose that the automorphisms $T_w : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$, $w \in W$ admit extensions to automorphisms $T_w : \mathcal{A} \rightarrow \mathcal{A}$ of a \mathfrak{g} -admissible algebra \mathcal{A} . Such an extension is uniquely determined by automorphisms $T_i : \mathcal{A} \rightarrow \mathcal{A}$, defined for all simple positive roots α_i , which extend the automorphisms $T_i : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$, see Section 2.1 and satisfy braid group relations, related to \mathfrak{g} :

$$(6.1) \quad \underbrace{T_i T_j T_i \cdots}_{m_{i,j}} = \underbrace{T_j T_i T_j \cdots}_{m_{i,j}}, \quad i \neq j,$$

where $m_{i,j} = 2$, if $a_{i,j} = 0$; $m_{i,j} = 3$, if $a_{i,j}a_{j,i} = 1$; $m_{i,j} = 4$, if $a_{i,j}a_{j,i} = 2$; $m_{i,j} = 6$, if $a_{i,j}a_{j,i} = 3$; and $m_{i,j} = \infty$, if $a_{i,j}a_{j,i} > 3$. Having (6.1), for any $w \in W$ we define the automorphism $T_w : \mathcal{A} \rightarrow \mathcal{A}$ by the relation $T_w = T_{i_1} \cdots T_{i_k}$, where $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_k}}$ is a

reduced decomposition of w . Then the elements T_w do not depend on a choice of the reduced decomposition and satisfy the relations

$$(6.2) \quad T_{w'w} = T_{w'} \cdot T_w, \quad \text{if } l(w'w) = l(w') + l(w).$$

The automorphisms T_i (and T_w) admit unique extensions to automorphisms of the algebra \mathcal{A}' , satisfying (6.1). We denote them by the same symbols.

For any maximal nilpotent subalgebra $\mathfrak{m} = \mathfrak{n}^w$, where $w \in W$, and a root α , such that the root vector e_α is a simple positive root vector of \mathfrak{m} , we have the following relations:

$$(6.3) \quad q_{\alpha, \mathfrak{m}} = T_w q_{w^{-1}(\alpha), \mathfrak{n}} T_w^{-1}.$$

For each $\alpha_i \in \Pi$ define operators $\check{q}_i : \mathcal{A}'/\mathcal{A}'\mathfrak{n} \rightarrow \mathcal{A}'/\mathcal{A}'\mathfrak{n}$ by the relations

$$(6.4) \quad \check{q}_i = q_{s_{\alpha_i}} \cdot T_i.$$

In (6.4), we understand $q_{s_{\alpha_i}} \equiv q_{\alpha_i, \mathfrak{n}}$ as maps $q_{\alpha_i, \mathfrak{n}} : \mathcal{A}'/\mathcal{A}'\mathfrak{n}^{s_{\alpha_i}} \rightarrow \mathcal{A}'/\mathcal{A}'\mathfrak{n}$, given by the relations (4.9)–(4.11). With the same agreement for any $w \in W$ we define operators $\check{q}_w : \mathcal{A}'/\mathcal{A}'\mathfrak{n} \rightarrow \mathcal{A}'/\mathcal{A}'\mathfrak{n}$ by the relations

$$(6.5) \quad \check{q}_w = q_w \cdot T_w.$$

The relation (6.3), Proposition 4.8 and its analog for the maps $\check{q}_{\bar{w}, \mathfrak{m}}$ imply

Proposition 6.1. *Operators \check{q}_i satisfy the braid group relations*

$$(6.6) \quad \underbrace{\check{q}_i \check{q}_j \cdots}_{m_{i,j}} = \underbrace{\check{q}_j \check{q}_i \cdots}_{m_{i,j}}, \quad i \neq j.$$

In other words, for any reduced decomposition $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_m}}$ we have equalities $\check{q}_w = \check{q}_{i_1} \cdots \check{q}_{i_m}$, so

$$(6.7) \quad \check{q}_{w'w} = \check{q}_{w'} \check{q}_w \quad \text{if } l(w'w) = l(w') + l(w).$$

Proof. Let $w', w \in W$ and $l(w'w) = l(w') + l(w)$. We have by (4.27) and (6.3)

$$\begin{aligned} \check{q}_{w'w} &= q_{w'w, \mathfrak{n}} T_{w'w} = q_{w', \mathfrak{n}} q_{w, \mathfrak{n}^{w'}} T_{w'w} \\ &= q_{w', \mathfrak{n}} (T_{w'} q_{w, \mathfrak{n}} T_{w'}^{-1}) T_{w'w} = q_{w', \mathfrak{n}} T_{w'} q_{w, \mathfrak{n}} T_w = \check{q}_{w'} \check{q}_w. \end{aligned}$$

Thus we have the relations (6.7), which are equivalent to the braid group relations. \square

For any $w \in W$ denote by $w \circ$ the shifted action of w in \mathfrak{h}^* :

$$w \circ \mu = w(\mu + \rho) - \rho.$$

It induces the shifted action by automorphisms of W in D , characterized by the relations

$$(6.8) \quad w \circ h_\alpha = h_{w(\alpha)} + \langle h_\alpha, w(\rho) - \rho \rangle.$$

Theorem 4 and Proposition 5.1 imply

Proposition 6.2. *For any $x \in \mathcal{A}'/\mathcal{A}'\mathfrak{n}$, $y \in Z^n(\mathcal{A})$ and $d \in D$ we have*

$$(6.9) \quad \begin{aligned} \check{q}_i([d, x]) &= [s_{\alpha_i}(d), \check{q}_i(x)], & \check{q}_i(xd) &= \check{q}_i(x) \cdot (s_{\alpha_i} \circ d), \\ \check{q}_i(xy) &= \check{q}_i(x) \cdot \check{q}_i(y). \end{aligned}$$

Corollary 6.3.

- (i) *The restriction of operators \check{q}_i to $Z^n(\mathcal{A})$ defines an automorphism of $Z^n(\mathcal{A})$, satisfying (6.9).*

(ii) For any $x \in Z^n(\mathcal{A})$ we have

$$\check{q}_i^2(x) = (h_{\alpha_i} + 1)^{-1} T_i^2(x) (h_{\alpha_i} + 1).$$

Proof. The statement (i) of the Corollary is a direct consequence of Proposition 6.2. The statement (ii) follows from Proposition 4.2 and Proposition 6. Namely, for any $x \in Z^n(\mathcal{A})$ we have by Proposition 4.2

$$\check{q}_i^2(x) = q_{\alpha_i, n} T_i q_{\alpha_i, n} T_i(x) = q_{\alpha_i, n} q_{-\alpha_i, n} s_{\alpha_i}(T_i^2(x)) = (h_{\alpha_i} + 1)^{-1} T_i^2(x) (h_{\alpha_i} + 1).$$

□

Clearly, all the statements of Proposition 6.2 remain valid for all operators \check{q}_w , $w \in W$. The properties (6.9), (7.16) look as

$$(6.10) \quad \begin{aligned} \check{q}_w([h, x]) &= [w(h), \check{q}_w(x)], & \check{q}_w(xd) &= \check{q}_w(x) \cdot (w \circ d), \\ \check{q}_w(xy) &= \check{q}_w(x) \cdot \check{q}_w(y) \end{aligned}$$

6.2. Calculation of $\check{q}_i(z_v)$. Denote by I_{n_-} the image of the right ideal $n_- \mathcal{A}'$ in $\mathcal{A}'/\mathcal{A}'n_-$:

$$I_{n_-} = (n_- \mathcal{A}' + \mathcal{A}' n_-) / \mathcal{A}' n_-.$$

Lemma 6.4. For any $\alpha_i \in \Pi$ we have an inclusion

$$(6.11) \quad \check{q}_i(I_{n_-}) \subset I_{n_-}.$$

Proof. We have to prove that for any $\gamma \in \Delta_+$ and $x \in \mathcal{A}'$

$$(6.12) \quad q_{\alpha_i}(T_i(e_{-\gamma}x)) = e_{-\mu}y$$

for some $\mu \in \Delta_+$ and $y \in \mathcal{A}'/\mathcal{A}'n_-$. If $\gamma \neq \alpha_i$ then

$$q_{\alpha_i}(T_i(e_{-\gamma}x)) = q_{\alpha_i}(e_{-\gamma'} \cdot T_i(x)),$$

where $\gamma' = s_{\alpha_i}(\gamma) \in \Delta_+ \setminus \alpha_i$, and the statement of the lemma follows from the invariance of the subalgebra $n_-(\alpha_i)$ with respect to the action of \hat{e}_{α_i} . Here $n_-(\alpha_i)$ is generated by root vectors $e_{-\gamma}$, where $\gamma \in \Delta_+ \setminus \alpha_i$.

If $\gamma = \alpha_i$ then the right hand side of (6.12) vanishes due to (4.3), (i). □

Suppose that an ad -invariant generating subspace \mathcal{V} of a \mathfrak{g} -admissible algebra \mathcal{A} is invariant with respect to the action of the automorphisms T_i , $T_i(\mathcal{V}) = \mathcal{V}$. Suppose that \mathcal{A} satisfies the highest weight (HW) condition. With these assumptions we calculate the elements $\check{q}_i(z_v)$, where $v \in \mathcal{V}$ and z_v are the generators of $Z^n(\mathcal{A})$, defined in Section 3.4.

Keep the notation of Section 3.4 and Section 3.5.

Proposition 6.5. Assume that \mathcal{A} satisfies the HW condition. Then

$$\check{q}_i(z_v) = z_{C_{\alpha_i}^{(2)}[-\rho](1 \otimes T_i(v))}.$$

The operators $C_{\pm\alpha}^{(2)}[\lambda]$ are defined in Section 3.5.

Proof. Assume that \mathcal{A} satisfies the HW condition. By definition of the elements z_v , we have $z_v = v \pmod{I_{n_-}}$. Lemma 6.4 then implies that

$$\check{q}_i(z_v) = \check{q}_i(v) \pmod{I_{n_-}}.$$

We have

$$\begin{aligned}
\check{q}_i(z_v) &= \check{q}_i(v) \mod I_{\mathbf{n}_-} \\
&= \sum_{n \geq 0} \frac{(-1)^n}{n!} \hat{e}_\alpha^n(v) e_{-\alpha}^n g_{\alpha,n} \mod I_{\mathbf{n}_-} \\
&= \sum_{n \geq 0} \frac{1}{n!} \hat{e}_{-\alpha}^n \hat{e}_\alpha^n(v) g_{\alpha,n} \mod I_{\mathbf{n}_-} \\
&= \sum_{n \geq 0} \frac{(-1)^n}{n!} \frac{(-1)^n}{(\hat{h}_\alpha - h_\alpha)(\hat{h}_\alpha - h_\alpha + 1) \cdots (\hat{h}_\alpha - h_\alpha + n - 1)} \hat{e}_{-\alpha}^n \hat{e}_\alpha^n(v) \mod I_{\mathbf{n}_-}.
\end{aligned}$$

This is precisely the statement of the proposition. \square

Remark. Let $w = s_{\alpha_1} \cdots s_{\alpha_n}$ be a reduced decomposition of an element $w \in W$. Let $\gamma_1, \dots, \gamma_n$ be the corresponding sequence of positive roots: $\gamma_1 = \alpha_1$, $\gamma_2 = s_{\alpha_1}(\alpha_2), \dots$. Then the properties of the maps \check{q}_i , see Proposition 6.2, imply the following relation

$$\check{q}_w(z_v) = z_{C_{\gamma_1}^{(2)}[-\rho] \cdots C_{\gamma_n}^{(2)}[-\rho](1 \otimes T_w(v))}.$$

6.3. Calculation of $\check{q}_i(z'_v)$. In this Section we assume that \mathfrak{g} is an arbitrary contragredient Lie algebra of finite growth and the generating subspace \mathcal{V} of a \mathfrak{g} -admissible algebra \mathcal{A} is invariant with respect to the action of the automorphisms T_i . With this assumption we calculate the elements $\check{q}_i(z'_v)$, where $v \in \mathcal{V}$ and z'_v are the generators of $Z^n(\mathcal{A})$, defined in Section 3.4.

Proposition 6.6. *Assume that the element $z'_v \in Z^n(\mathcal{A})$ is defined. Then the element $\check{q}_i(z'_v)$ is also defined and is given by the relation*

$$(6.13) \quad \check{q}_i(z'_v) = z'_{B_{-\alpha_i}^{(2)}[\rho](1 \otimes T_i(v))}.$$

The proof of Proposition 6.6 is based on the following Lemma.

Let α be a simple positive root, $\mathbf{n}_\pm(\alpha)$ subalgebras of \mathbf{n}_\pm , generated by all root vectors except $e_{\pm\alpha}$.

Lemma 6.7. *Suppose that the element z'_v exists. Then it admits a presentation*

$$(6.14) \quad z'_v = q_\alpha(v) + \sum_{n \geq 0, j} e_{-\alpha}^n v_{j,n} d_{j,n} f_{j,n},$$

where $v_{j,n} \in \mathcal{V}$, $d_{j,n} \in D$, $f_{j,n} \in \mathbf{n}_-(\alpha)U(\mathbf{n}_-(\alpha))$.

Proof of Lemma. Using the PBW theorem, present z'_v in a form

$$z'_v = x + y,$$

where

$$\begin{aligned}
x &= v + \sum_k v_k d_k e_{-\alpha}^k, & v_k &\in \mathcal{V}, d_k \in D, \\
y &= \sum_{n \geq 0, j} \tilde{v}_{j,n} e_{-\alpha}^n d_{j,n} f_{j,n}, & \tilde{v}_{j,n} &\in \mathcal{V}, d_{j,n} \in D, f_{j,n} \in \mathbf{n}_-(\alpha)U(\mathbf{n}_-(\alpha)).
\end{aligned}$$

By definition, z'_w satisfies the equation $[e_\alpha, z'_w] = 0 \pmod{\mathcal{A}'\mathfrak{n}}$. Since the algebra $\mathfrak{n}_-(\alpha)$ is invariant with respect to the adjoint action of e_α , the commutator $[e_\alpha, y]$ is an element z of the same kind, so we have two equations

$$[e_\alpha, x] = 0 \pmod{\mathcal{A}'\mathfrak{n}} \quad \text{and} \quad [e_\alpha, y] = 0 \pmod{\mathcal{A}'\mathfrak{n}}.$$

The first equation has a unique solution $x = q_\alpha(v)$. To finish the proof of the Lemma, we move all factors $e_{-\alpha}^n$ in the presentation of y to the left, using commutation relations in \mathcal{A} . \square

Proof of Proposition 6.6. The application of the automorphism T_i to the presentation (6.14) gives

$$(6.15) \quad T_i q_{\alpha_i, \mathfrak{n}}(u) = T_i q_{\alpha_i, \mathfrak{n}}(v) + \sum_{n \geq 0, j} e_{\alpha_i}^n \bar{v}_{j,n} \bar{d}_{j,n} \bar{f}_{j,n},$$

where $\bar{v}_{j,n} = T_i(\tilde{v}_{j,n}) \in \mathcal{V}$, $\bar{f}_{j,n} = T_i(\tilde{f}_{j,n}) \in \mathfrak{n}_-(\alpha_i)U(\mathfrak{n}_-(\alpha_i))$ and $\bar{d}_{j,n} = T_i(\tilde{d}_{j,n}) \in D$. Now we apply the map $q_{\alpha_i, \mathfrak{n}}$ to both sides of (6.15). The images of the terms in the last sum with $n > 0$ vanish, since $q_{\alpha_i, \mathfrak{n}}(e_{\alpha_i} x) = 0$ for any $x \in \mathcal{A}'$ by (4.3), (iv). The images of the terms in the last sum with $n = 0$ do not contribute to the 'leading term', since the algebra $\mathfrak{n}_-(\alpha_i)$ is invariant with respect to the adjoint action of \hat{e}_{α_i} . We obtained the statement:

Lemma 6.8. *The leading term of $\check{q}_i(z'_v)$ is equal to the leading term of $q_{\alpha_i, \mathfrak{n}} T_i q_{\alpha_i}(v)$.*

Let us compute the leading term of $q_{\alpha_i, \mathfrak{n}} T_i q_{\alpha_i}(v)$. We have

$$\begin{aligned} q_{\alpha_i, \mathfrak{n}} T_i q_{\alpha_i}(v) &= q_{\alpha_i, \mathfrak{n}} T_i \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{e}_{\alpha_i}^n(v) e_{-\alpha_i}^n g_{n, \alpha_i} \right) \\ &= q_{\alpha_i, \mathfrak{n}} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \hat{e}_{-\alpha_i}^n(T_i(v)) e_{\alpha_i}^n ((h_{\alpha_i})(h_{\alpha_i} + 1) \cdots (h_{\alpha_i} + n - 1))^{-1} \right) \\ &= q_{\alpha_i, \mathfrak{n}} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \hat{e}_{-\alpha_i}^n(T_i(v)) e_{\alpha_i}^n \right) ((h_{\alpha_i} + 2)(h_{\alpha_i} + 3) \cdots (h_{\alpha_i} + n + 1))^{-1}, \end{aligned}$$

by the property (4.3), (iii). Since $q_{\alpha_i, \mathfrak{n}}(e_{\alpha_i} x) = 0$ for any $x \in \mathcal{A}'$, we get further

$$q_{\alpha_i, \mathfrak{n}} T_i q_{\alpha_i}(v) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} q_{\alpha_i, \mathfrak{n}}(\hat{e}_{\alpha_i}^n \hat{e}_{-\alpha_i}^n(T_i(v))) ((h_{\alpha_i} + 2) \cdots (h_{\alpha_i} + n + 1))^{-1}.$$

Since $\hat{e}_{\alpha_i}^n \hat{e}_{-\alpha_i}^n(T_i(v))$ belongs to \mathcal{V} , the leading term of $q_{\alpha_i, \mathfrak{n}}(\hat{e}_{\alpha_i}^n \hat{e}_{-\alpha_i}^n(T_i(v)))$ is equal to $\hat{e}_{\alpha_i}^n \hat{e}_{-\alpha_i}^n(T_i(v))$ and the leading term of $q_{\alpha_i, \mathfrak{n}} T_i q_{\alpha_i}(v)$ is equal to the sum

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{e}_{\alpha_i}^n \hat{e}_{-\alpha_i}^n(T_i(v)) ((h_{\alpha_i} + 2) \cdots (h_{\alpha_i} + n + 1))^{-1},$$

which can be written as $B_{-\alpha_i}^{(2)}[\rho](1 \otimes T_i(v))$. \square

As well as in the previous section, for a reduced decomposition $w = s_{\alpha_1} \cdots s_{\alpha_n}$ of an element $w \in W$ and a corresponding sequence of positive roots: $\gamma_1 = \alpha_1$, $\gamma_2 = s_{\alpha_1}(\alpha_2), \dots$ we have, assuming the existence of z'_v ,

$$\check{q}_w(z'_v) = z'_{B_{-\gamma_1}^{(2)}[\rho] \cdots B_{\gamma_n}^{(2)}[\rho](1 \otimes T_w(v))}.$$

7. MICKELSSON ALGEBRA $Z_{\mathfrak{n}_-}(\mathcal{A})$

In this section we collect the results of the previous section for a Mickelsson algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$, see the definition below. This algebra deserves a special attention, since it acts on \mathfrak{n}_- -coinvariants, which are sometimes more convenient than \mathfrak{n} -invariants. If a \mathfrak{g} -admissible algebra \mathcal{A} admits an antiinvolution, which extends the Cartan antiinvolution ${}^t : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$, then all the results of this section can be obtained by an application of this antiinvolution to corresponding results of the previous sections.

7.1. Algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$ and related structures. For any associative algebra \mathcal{A} , which contains $U(\mathfrak{g})$, we define Mickelsson algebras $S_{\mathfrak{n}_-}(\mathcal{A})$ and $Z_{\mathfrak{n}_-}(\mathcal{A})$ as the quotients

$$S_{\mathfrak{n}_-}(\mathcal{A}) = \mathfrak{n}_-\mathcal{A} \backslash \text{Nr}(\mathfrak{n}_-\mathcal{A}), \quad Z_{\mathfrak{n}_-}(\mathcal{A}) = \mathfrak{n}_-\mathcal{A}' \backslash \text{Nr}(\mathfrak{n}_-\mathcal{A}'),$$

where $\text{Nr}(\mathfrak{n}_-\mathcal{A})$ ($\text{Nr}(\mathfrak{n}_-\mathcal{A}')$) is the normalizer of the right ideal $\mathfrak{n}_-\mathcal{A}$ ($\mathfrak{n}_-\mathcal{A}'$). The algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$ is a localization of the algebra $S_{\mathfrak{n}_-}(\mathcal{A})$: $Z_{\mathfrak{n}_-}(\mathcal{A}) = D \otimes_{U(\mathfrak{h})} S_{\mathfrak{n}_-}(\mathcal{A})$.

Alternatively, the Mickelsson algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$ can be described as a subspace of \mathfrak{n}_- -invariants in a right $U(\mathfrak{g})$ -module $\tilde{M}_{\mathfrak{n}_-}(\mathcal{A}') = \mathfrak{n}_-\mathcal{A}' \backslash \mathcal{A}'$:

$$Z_{\mathfrak{n}_-}(\mathcal{A}) = \left(\tilde{M}_{\mathfrak{n}_-}(\mathcal{A}') \right)^{\mathfrak{n}_-} = \{m \in \tilde{M}_{\mathfrak{n}_-}(\mathcal{A}') \mid m\mathfrak{n}_- = 0\}.$$

As well as $S^n(\mathcal{A})$, the space $S_{\mathfrak{n}_-}(\mathcal{A})$ is an associative algebra, containing $U(\mathfrak{h})$, and for any left \mathcal{A} -module M , the space $M_{\mathfrak{n}_-} = M/\mathfrak{n}_-M$ of \mathfrak{n}_- -coinvariants is a $S_{\mathfrak{n}_-}(\mathcal{A})$ -module, see Proposition 3.1. The algebra $Z_{\mathfrak{n}_-}$ acts in the space $M_{\mathfrak{n}_-}$ of \mathfrak{n}_- -coinvariants of any left \mathcal{A}' -module M .

Suppose that a \mathfrak{g} -admissible algebra \mathcal{A} satisfies the additional *local lowest weight* condition

(LW) For any $v \in \mathcal{V}$ the adjoint action of elements $x \in U(\mathfrak{n}_-)_{\mu}$ on v is nontrivial, $\hat{x}(v) \neq 0$, only for a finite number of $\mu \in \mathfrak{h}^*$.

Then the quotient $\tilde{M}_{\mathfrak{n}_-}(\mathcal{A}') = \mathfrak{n}_-\mathcal{A}' \backslash \mathcal{A}'n$ has a structure of a right $F_{\mathfrak{g},\mathfrak{n}}$ -module, extending the action of \mathcal{A}' by the right multiplication. In particular, the extremal projector P acts in the right $F_{\mathfrak{g},\mathfrak{n}}$ -module $\tilde{M}_{\mathfrak{n}_-}(\mathcal{A}')$. Denote the corresponding operator by $\bar{p} \in \text{End } \tilde{M}_{\mathfrak{n}_-}(\mathcal{A}')$, see Section 2.2.

The properties of the extremal projectors imply the relation

$$(7.1) \quad Z_{\mathfrak{n}_-}(\mathcal{A}) = \text{Im } \bar{p} \subset \tilde{M}_{\mathfrak{n}_-}(\mathcal{A}').$$

Equip the double coset space ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$, see (3.6), with a multiplication $\circ : {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} \otimes {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} \rightarrow {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$:

$$(7.2) \quad a \circ b = aPb \stackrel{\text{def}}{=} \bar{p}(a) \cdot b.$$

We also call the double coset space ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$, equipped with the operation (7.2), the *double coset algebra* ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$. In a case, when both conditions (HW) and (LW) are satisfied, the multiplication rules (3.7) and (7.2) coincide.

Define linear maps $\phi_- : Z_{\mathfrak{n}_-}(\mathcal{A}) \rightarrow {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ and $\psi_- : {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} \rightarrow Z_{\mathfrak{n}_-}(\mathcal{A})$ by the rules

$$(7.3) \quad \phi_-(x) = x \mod \mathcal{A}'\mathfrak{n}, \quad \psi^+(y) = \bar{p}(y), \quad x \in Z_{\mathfrak{n}_-}(\mathcal{A}), y \in {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}.$$

Proposition 7.1. *Assume that a \mathfrak{g} -admissible algebra \mathcal{A} satisfies the condition (LW). Then*

- (i) *The operation (3.7) equips ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ with a structure of an associative algebra.*

- (ii) The linear maps ϕ_- and ψ_- are inverse to each other and establish an isomorphism of the algebras $Z_{\mathfrak{n}_-}(\mathcal{A})$ and ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}_-}$.

We have the 'lowest weight counterparts' of Propositions 3.3 and 3.4.

Proposition 7.2. *Let \mathcal{A} be a \mathfrak{g} -admissible algebra satisfying the condition (LW). Then*

- (i) *Each element of the double coset algebra ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}_-}$ can be uniquely presented in a form $x = \sum_i d_i v_i$, where $d_i \in D$, $v_i \in \mathcal{V}$, so that ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}_-}$ is a free left (and right) D -module, isomorphic to $D \otimes \mathcal{V}$.*
- (ii) *For each $v \in \mathcal{V}$ there exists a unique element $\tilde{z}_v \in Z_{\mathfrak{n}_-}(\mathcal{A})$ of the form*

$$(7.4) \quad \tilde{z}_v = v + \sum_{i=1, \dots, k} v_i e_i d_i, \quad e_i \in \mathfrak{n}U(\mathfrak{n}), \quad d_i \in D, \quad v_i \in \mathcal{V},$$

so that the algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$ is a free left (and right) D -module, generated by the elements \tilde{z}_v . The element \tilde{z}_v is equal to $\bar{\mathfrak{p}}(v)$.

- (iii) *For each $v \in \mathcal{V}$ there exists at most one element $\tilde{z}'_v \in Z_{\mathfrak{n}_-}(\mathcal{A})$ of the form*

$$(7.5) \quad \tilde{z}'_v = v + \sum_j e_j v_j d_j, \quad e_j \in \mathfrak{n}U(\mathfrak{n}), \quad d_j \in D, \quad v_j \in \mathcal{V}.$$

Next, we have analogs of Theorems 1 and 2:

Theorem 5. *Let \mathfrak{g} be reductive and finite-dimensional, \mathcal{A} a \mathfrak{g} -admissible algebra with a generating subspace \mathcal{V} . Then for any $v \in \mathcal{V}$*

- (i) *there exists a unique element $\tilde{z}'_v \in Z_{\mathfrak{n}_-}(\mathcal{A})$ of the form (7.5). The algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$ is generated by the elements \tilde{z}'_v as a free left (and right) D -module;*
- (ii) *we have the following equality in $Z_{\mathfrak{n}_-}(\mathcal{A})$*

$$(7.6) \quad \tilde{z}_v = \tilde{z}'_{B_-^{(2)}[\rho](1 \otimes v)}.$$

Here $\tilde{z}_{d \otimes v} = d \cdot \tilde{z}_v$, $\tilde{z}'_{d \otimes v} = d \cdot \tilde{z}'_v$, $\tilde{z}_{v \otimes d} = \tilde{z}_v \cdot d$, $\tilde{z}'_{v \otimes d} = \tilde{z}'_v \cdot d$.

7.2. Zhelobenko maps. For any real root α , the relations

$$\begin{aligned} \tilde{q}_\alpha(x) &= \sum_{n=0}^{\infty} \frac{1}{(n)!} g_{n,\alpha} \cdot e_\alpha^n \cdot \hat{e}_{-\alpha}^n(x) \quad \text{mod } \mathfrak{n}_{-\alpha}\mathcal{A}', \quad x \in \mathcal{A}, \\ \tilde{q}_\alpha(dx) &= \tau_\alpha(d) \tilde{q}_\alpha(x), \quad d \in D \end{aligned}$$

define a map $\tilde{q}_\alpha : \mathcal{A}' \rightarrow \mathfrak{n}_{-\alpha}\mathcal{A}' \setminus \mathcal{A}'$, such that for any $x \in \mathcal{A}'$

$$(7.7) \quad \tilde{q}_\alpha(e_\alpha x) = 0, \quad \tilde{q}_\alpha(x e_{-\alpha}) = \tilde{q}_\alpha(x) e_{-\alpha} = 0.$$

Here $g_{n,\alpha}$ is given by (4.2), $\mathfrak{n}_{\pm\alpha} = \mathbb{C}e_{\pm\alpha}$. We have analogs of Propositions 4.1, 4.2 and Theorem 3.

Theorem 6.

- (i) *The map \tilde{q}_α defines an isomorphism of algebras $\tilde{q}_\alpha : {}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}} \rightarrow Z_{\mathfrak{n}_{-\alpha}}(\mathcal{A})$, such that for any $d \in D$,*

$$[d, \tilde{q}_\alpha(x)] = \tilde{q}_\alpha([d, x]), \quad \tilde{q}_\alpha(dx) = \tau_\alpha(d) \tilde{q}_\alpha(x).$$

- (ii) *For any $x \in Z_{\mathfrak{n}_\alpha}(\mathcal{A})$ and $y \in Z_{\mathfrak{n}_{-\alpha}}(\mathcal{A})$ we have*

$$(7.8) \quad \tilde{q}_{-\alpha} \tilde{q}_\alpha(x) = (h_\alpha + 1)x(h_\alpha + 1)^{-1}, \quad \tilde{q}_\alpha \tilde{q}_{-\alpha}(y) = (h_\alpha + 1)^{-1}y(h_\alpha + 1).$$

Theorem 6 implies the equality

$$\tilde{q}_\alpha(xy) = \tilde{q}(x)\tilde{q}(y) \quad \text{for any } x \in \text{Nr}(\mathfrak{n}_\alpha \mathcal{A}'), y \in \mathcal{A}'.$$

For a maximal nilpotent subalgebra \mathfrak{m} of \mathfrak{g} , such that e_α is a simple positive root vector of \mathfrak{m} , define the linear map $\tilde{q}_{\alpha,\mathfrak{m}}^{(k)} : \mathcal{A}' \rightarrow \mathfrak{m}_- \mathcal{A}' \setminus \mathcal{A}'$ by the prescriptions

$$(7.9) \quad \tilde{q}_{\alpha,\mathfrak{m}}(x) = \sum_{n=0}^{\infty} \frac{1}{(n)!} g_{n,\alpha} \cdot e_\alpha^n \cdot \hat{e}_{-\alpha}^n(x) \quad \text{mod } \mathfrak{m}_- \mathcal{A}', \quad x \in \mathcal{A},$$

$$(7.10) \quad \tilde{q}_{\alpha,\mathfrak{m}}(dx) = \tau_\alpha(d)\tilde{q}_\alpha(x), \quad d \in D.$$

The assignment $\tilde{q}_{\alpha,\mathfrak{m}}$ satisfies the relation $\tilde{q}_{\alpha,\mathfrak{m}}(\mathfrak{m}_-^{s_\alpha} \mathcal{A}') = 0$ and determines the map

$$\tilde{q}_{\alpha,\mathfrak{m}} : \mathfrak{m}_-^{s_\alpha} \mathcal{A}' \setminus \mathcal{A}' \rightarrow \mathfrak{m}_- \mathcal{A}' \setminus \mathcal{A}'.$$

We have, see Proposition 4.4 and Theorem 4:

Proposition 7.3.

(i) The map $\tilde{q}_{\alpha,\mathfrak{m}}$ transforms $\text{Nr}(\mathfrak{m}_-^{s_\alpha} \mathcal{A}')$ to the Mickelsson algebra $Z_{\mathfrak{m}_-}(\mathcal{A})$ and

$$\tilde{q}_{\alpha,\mathfrak{m}}(zu) = \tilde{q}_{\alpha,\mathfrak{m}}(z)\tilde{q}_{\alpha,\mathfrak{m}}(u) \quad \text{for any } z \in \text{Nr}(\mathfrak{m}_-^{s_\alpha} \mathcal{A}'), u \in \mathcal{A}'.$$

(ii) The restriction of the map $\tilde{q}_{\alpha,\mathfrak{m}}$ to the normalizer $\text{Nr}(\mathfrak{m}_-^{s_\alpha} \mathcal{A}')$ defines an isomorphism of the algebras $Z_{\mathfrak{m}_-^{s_\alpha}}(\mathcal{A})$ and $Z_{\mathfrak{m}_-}(\mathcal{A})$, satisfying the relations

$$(7.11) \quad [d, \tilde{q}_{\alpha,\mathfrak{m}}(x)] = \tilde{q}_{\alpha,\mathfrak{m}}([d, x]), \quad \tilde{q}_{\alpha,\mathfrak{m}}(xd) = \tilde{q}_{\alpha,\mathfrak{m}}(x)\tau_\alpha(d), \quad d \in D.$$

For any $w \in W$, satisfying the condition (4.16), and its reduced decomposition (4.17) we define a map $\tilde{q}_{w,\mathfrak{m}} : \mathfrak{m}_-^w \mathcal{A}' \setminus \mathcal{A}' \rightarrow \mathfrak{m}_- \mathcal{A}' \setminus \mathcal{A}'$ by the relation

$$\tilde{q}_{w,\mathfrak{m}} = \tilde{q}_{\gamma_1, \mathfrak{m}_1} \cdot \tilde{q}_{\gamma_2, \mathfrak{m}_2} \cdot \dots \cdot \tilde{q}_{\gamma_n, \mathfrak{m}_n},$$

where positive roots γ_k and maximal nilpotent subalgebras \mathfrak{m}_k are defined by the prescriptions (4.18)–(4.19). This map does not depend on a choice of the reduced decomposition of w and satisfies the relations

$$(7.12) \quad \tilde{q}_{w'w, \mathfrak{m}} = \tilde{q}_{w', \mathfrak{m}} \tilde{q}_{w, \mathfrak{m}^{w'}} \quad \text{if } l(w'w) = l(w') + l(w),$$

$$(7.13) \quad [h, \tilde{q}_{w,\mathfrak{m}}(x)] = \tilde{q}_{w,\mathfrak{m}}([h, x]), \quad \tilde{q}_{w,\mathfrak{m}}(dx) = \tau_{w'(\rho) - w'w(\rho)}(d) \cdot \tilde{q}_{w,\mathfrak{m}}(x)$$

for any $x \in \mathcal{A}'$, $h \in \mathfrak{h}$ and $d \in D$.

The restriction of $\tilde{q}_{w,\mathfrak{m}}$ to the normalizer $\text{Nr}(\mathfrak{m}_-^w \mathcal{A}')$ defines an isomorphism of the algebras $Z_{\mathfrak{m}_-^w}(\mathcal{A})$ and $Z_{\mathfrak{m}_-}(\mathcal{A})$, satisfying (7.13). We denote $\tilde{q}_w \equiv \tilde{q}_{w,\mathfrak{n}}$.

The following counterpart of Proposition 4.6 is valid.

Proposition 7.4. Let \mathfrak{g} be a finite-dimensional reductive Lie algebra. Let w_0 be the longest element of the Weyl group W . Then

(i) The map \tilde{q}_{w_0} defines an isomorphism of the algebras $\mathfrak{n}_- \mathcal{A}_{\mathfrak{n}}$ and $Z_{\mathfrak{n}_-}(\mathcal{A})$, such that for any $x \in \mathfrak{n}_- \mathcal{A}_{\mathfrak{n}}$, $d \in D$ and $v \in \mathcal{V}$

$$(7.14) \quad [d, \tilde{q}_{w_0}(x)] = \tilde{q}_{w_0}([d, x]), \quad \tilde{q}_{w_0}(dx) = \tau_{\rho - w_0(\rho)}(d)\tilde{q}_{w_0}(x),$$

$$\tilde{q}_{w_0}(v) = \tilde{z}'_v.$$

(ii) The restriction of \tilde{q}_{w_0} to the normalizer $\text{Nr}(\mathfrak{n}_- \mathcal{A}')$ defines an isomorphism of the algebras $Z_{\mathfrak{n}_-}(\mathcal{A})$ and $Z_{\mathfrak{n}_-}(\mathcal{A})$, satisfying (7.14), such that

$$\tilde{q}_{w_0}(\tilde{z}_{\mathfrak{n}_-, v}) = \tilde{z}'_{\mathfrak{n}_-, v}, \quad v \in \mathcal{V}.$$

Here $\tilde{z}_{\mathbf{n},v}$ are the generators of the Mickelsson algebra $Z_{\mathbf{n}}(\mathcal{A})$ of the 'first type', see Proposition 7.2 (i), $\tilde{z}'_{\mathbf{n}_-,v}$ are the generators of the Mickelsson algebra $Z_{\mathbf{n}_-}(\mathcal{A})$ of the 'second type', see Proposition 7.2 (ii).

7.3. Braid group action. Keep the notation of Section 6. We suppose again that the automorphisms $T_i : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$, $i = 1, \dots, r$, see Section 2.1 admit extensions to automorphisms $T_i : \mathcal{A} \rightarrow \mathcal{A}$ of a \mathfrak{g} -admissible algebra \mathcal{A} , satisfying the braid group relations (6.1).

Then for each maximal nilpotent subalgebra $\mathfrak{m} = \mathfrak{n}^w$, where $w \in W$, and a root α , such that the root vector e_α is a simple positive root vector of \mathfrak{m} , we have the relations:

$$(7.15) \quad \check{q}_{\alpha, \mathfrak{m}} = T_w \check{q}_{w^{-1}(\alpha), \mathfrak{n}} T_w^{-1}.$$

For each $\alpha_i \in \Pi$ define operators $\check{q}_i : \mathfrak{n}_- \mathcal{A}' \backslash \mathcal{A}' \rightarrow \mathfrak{n}_- \mathcal{A}' \backslash \mathcal{A}'$ and $\check{q}_w : \mathfrak{n}_- \mathcal{A}' \backslash \mathcal{A}' \rightarrow \mathfrak{n}_- \mathcal{A}' \backslash \mathcal{A}'$ as

$$\check{q}_i = \check{q}_{s_{\alpha_i}} \cdot T_i, \quad \check{q}_w = \check{q}_w \cdot T_w.$$

The operators \check{q}_i satisfy the braid group relations,

$$\underbrace{\check{q}_i \check{q}_j \cdots}_{m_{i,j}} = \underbrace{\check{q}_j \check{q}_i \cdots}_{m_{i,j}}, \quad i \neq j,$$

that is,

$$\check{q}_{w'w} = \check{q}_{w'} \check{q}_w, \quad \text{if } l(w'w) = l(w') + l(w).$$

For any $w \in W$, $x \in Z_{\mathbf{n}_-}(\mathcal{A})$, $y \in \mathfrak{n}_- \mathcal{A}' \backslash \mathcal{A}'$ and $d \in D$ we have by Proposition 7.3:

$$(7.16) \quad \begin{aligned} \check{q}_w([h, x]) &= [w(h), \check{q}_w(x)], & \check{q}_w(dx) &= (w \circ d) \cdot \check{q}_w(x), \\ \check{q}_w(xy) &= \check{q}_w(x) \cdot \check{q}_w(y). \end{aligned}$$

Corollary 7.5.

- (i) *The restriction of \check{q}_i to $Z_{\mathbf{n}_-}(\mathcal{A})$ defines an automorphism of $Z_{\mathbf{n}_-}(\mathcal{A})$, satisfying (7.16).*
- (ii) *For any $y \in Z_{\mathbf{n}_-}(\mathcal{A})$ we have*

$$\check{q}_i^2(y) = (h_{\alpha_i} + 1)^{-1} T_i^2(y) (h_{\alpha_i} + 1).$$

The following proposition describes the action of the automorphisms \check{q}_i on the canonical generators of the Mickelsson algebra $Z_{\mathbf{n}_-}(\mathcal{A})$.

Proposition 7.6.

- (i) *Assume that \mathcal{A} satisfies the LW condition. Then*

$$\check{q}_i(\tilde{z}_v) = \tilde{z}_{C_{-\alpha_i}^{(1)}[-\rho](T_i(v) \otimes 1)}.$$

- (ii) *Assume that the element $\tilde{z}'_v \in Z_{\mathbf{n}_-}(\mathcal{A})$ is defined. Then $\check{q}_i(\tilde{z}'_v)$ is also defined and is given by the relation*

$$\check{q}_i(\tilde{z}'_v) = \tilde{z}'_{B_{\alpha_i}^{(1)}[\rho](T_i(v) \otimes 1)}.$$

The operators $C_{\pm\alpha}^{(1)}[\lambda]$ and $B_{\pm\alpha}^{(1)}[\lambda]$ are defined in Section 3.5.

Remark. Let $w = s_{\alpha_1} \cdots s_{\alpha_n}$ be a reduced decomposition of an element $w \in W$. Let $\gamma_1, \dots, \gamma_n$ be the corresponding sequence of positive roots: $\gamma_1 = \alpha_1$, $\gamma_2 = s_{\alpha_1}(\alpha_2), \dots$

Then the properties of the maps \check{q}_i and \check{q}_i , see Proposition 6.2 imply the following relations

$$\check{q}_w(\tilde{z}_v) = \tilde{z}_{C_{-\gamma_1}^{(1)}[-\rho] \cdots C_{-\gamma_n}^{(1)}[-\rho](T_w(v) \otimes 1)} , \quad \check{q}_w(\tilde{z}'_v) = \tilde{z}'_{B_{\gamma_1}^{(1)}[\rho] \cdots B_{-\gamma_n}^{(1)}[\rho](T_w(v) \otimes 1)} .$$

8. STANDARD MODULES AND DYNAMICAL WEYL GROUP

8.1. Double coset space. Recall the notation ${}_n\mathcal{A}_n = {}_n\mathcal{A}' \backslash \mathcal{A}' / \mathcal{A}' n$ from Section 3.3.

Lemma 8.1. *The multiplication in \mathcal{A}' equips the double coset space ${}_n\mathcal{A}_n$ with the structure of the left $Z_{n-}(\mathcal{A})$ -module and the right $Z^n(\mathcal{A})$ -module.*

Proof follows from the definition of normalizers. \square

If \mathcal{A} satisfies the HW condition, Proposition 3.2 says that the double coset space ${}_n\mathcal{A}_n$ is a free right $Z^n(\mathcal{A})$ -module of rank one, generated by the class of 1. If \mathcal{A} satisfies the LW condition, Proposition 7.1 says that the double coset space ${}_n\mathcal{A}_n$ is a free left $Z_{n-}(\mathcal{A})$ -module of rank one, generated by the class of 1.

Lemma 6.4 says that the operators $\check{q}_i : \mathcal{A}' / \mathcal{A}' n \rightarrow \mathcal{A}' / \mathcal{A}' n$ and $\check{q}_i : {}_n\mathcal{A}' \backslash \mathcal{A}' \rightarrow {}_n\mathcal{A}' \backslash \mathcal{A}'$ correctly define operators in the double coset space ${}_n\mathcal{A}_n$. We denote them by the same symbol. According to definitions, they are given by the formulas

$$\begin{aligned} \check{q}_i(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{e}_{\alpha_i}^n(T_i(x)) \cdot e_{-\alpha_i}^n \cdot g_{n,\alpha_i} \quad \text{mod } {}_n\mathcal{A}' + \mathcal{A}' n, \\ \check{q}_i(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} g_{n,\alpha_i} \cdot e_{\alpha_i}^n \cdot \hat{e}_{-\alpha_i}^n(T_i(x)) \quad \text{mod } {}_n\mathcal{A}' + \mathcal{A}' n, \end{aligned}$$

where $g_{n,\alpha_i} = (h_{\alpha_i}(h_{\alpha_i} - 1) \cdots (h_{\alpha_i} - n + 1))^{-1}$. Equivalently,

$$(8.1) \quad \check{q}_i(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{e}_{-\alpha_i}^n \hat{e}_{\alpha_i}^n(T_i(x)) \cdot g_{n,\alpha_i} \quad \text{mod } {}_n\mathcal{A}' + \mathcal{A}' n,$$

$$(8.2) \quad \check{q}_i(x) = \sum_{n=0}^{\infty} \frac{1}{n!} g_{n,\alpha_i} \cdot \hat{e}_{\alpha_i}^n \hat{e}_{-\alpha_i}^n(T_i(x)) \quad \text{mod } {}_n\mathcal{A}' + \mathcal{A}' n.$$

They satisfy all the properties, mentioned in Proposition 6.1, Proposition 6.2 and Corollary 6.3.

In the notation of Section 3.5, the formulas (8.1) and (8.2) mean that for any $v \in \mathcal{V}$ we have the following equalities in ${}_n\mathcal{A}_n$:

$$(8.3) \quad \check{q}_i(v) = m \left(C_{\alpha_i}^{(2)}[-\rho](1 \otimes T_i(v)) \right), \quad \check{q}_i(v) = m \left(C_{-\alpha_i}^{(1)}[-\rho](T_i(v) \otimes 1) \right),$$

where $m : D \otimes \mathcal{V} \rightarrow \mathcal{A}'$ and $m : \mathcal{V} \otimes D \rightarrow \mathcal{A}'$ are the multiplication maps.

Let M be a module over an associative algebra U . Denote by ξ_M the corresponding homomorphism $\xi_M : U \rightarrow \text{End}(M)$. Let $T : U \rightarrow U$ be an automorphism of the algebra U . Denote by M^T the U -module M , conjugated by the automorphism T . It can be described as follows. M^T coincides with M as a vector space, while the map $\xi_{M^T} : U \rightarrow \text{End}(M) \equiv \text{End}(M^T)$ is

$$\xi_{M^T} = \xi_M \cdot T.$$

In this notation, Proposition 6.2 states the equivariance of the maps \check{q}_i and \check{q}_i :

Proposition 8.2.

(i) The map \check{q}_i , given by (8.1), is a morphism of the right $Z^n(\mathcal{A})$ -modules:

$$\check{q}_i : {}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n}} \rightarrow ({}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n}})^{\check{q}_i}.$$

(ii) The map \check{q}_i , given by (8.2), is a morphism of the left $Z_{\mathbf{n}_-}(\mathcal{A})$ -modules:

$$\check{q}_i : {}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n}} \rightarrow ({}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n}})^{\check{q}_i}.$$

The same statement holds for the operators \check{q}_w , defined as products of (8.1), and the operators \check{q}_w , defined as products of (8.2), where $w \in W$.

Let $\lambda \in \mathfrak{h}^*$ be a generic weight, that is, $\langle h_\alpha, \lambda \rangle \notin \mathbb{Z}$ for any $\alpha \in \Delta$. Then the following quotients of the double coset space are well defined:

$$(8.4) \quad \begin{aligned} {}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n},\lambda} &= {}_{\mathbf{n}_-}\mathcal{A}' \backslash \mathcal{A}' / \mathcal{A}' \cdot (\mathbf{n}, h - \langle h, \lambda \rangle)|_{h \in \mathfrak{h}}, \\ {}_{\lambda, \mathbf{n}_-}\mathcal{A}_{\mathbf{n}} &= ((h - \langle h, \lambda \rangle)|_{h \in \mathfrak{h}}, \mathbf{n}_-) \mathcal{A}' \backslash \mathcal{A}' / \mathcal{A}' \mathbf{n}. \end{aligned}$$

The space ${}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n},\lambda}$ is a left $Z_{\mathbf{n}_-}(\mathcal{A})$ -module, the space ${}_{\lambda, \mathbf{n}_-}\mathcal{A}_{\mathbf{n}}$ is a right $Z^n(\mathcal{A})$ -module.

Corollary 8.3. For a generic $\lambda \in \mathfrak{h}^*$,

(i) the map (8.1) defines a morphism of the right $Z^n(\mathcal{A})$ -modules:

$$\check{q}_{i,\lambda} : {}_{\lambda, \mathbf{n}_-}\mathcal{A}_{\mathbf{n}} \rightarrow ({}_{s_{\alpha_i} \circ \lambda, \mathbf{n}_-}\mathcal{A}_{\mathbf{n}})^{\check{q}_i};$$

(ii) the map (8.2) defines a morphism of the left $Z_{\mathbf{n}_-}(\mathcal{A})$ -modules:

$$\check{q}_{i,\lambda} : {}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n},\lambda} \rightarrow ({}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n}, s_{\alpha_i} \circ \lambda})^{\check{q}_i}.$$

Due to (8.3), we have

$$\check{q}_{i,\lambda}(v) = \hat{p}_{\alpha_i}[\lambda - \rho](T_i(v)), \quad \check{q}_{i,\lambda}(v) = \hat{p}_{-\alpha_i}[\lambda - \rho](T_i(v)),$$

that is,

$$\begin{aligned} \check{q}_{i,\lambda}(v) &= \sum_{n \geq 0} \frac{(-1)^n}{n!} \prod_{k=1}^n \left(\hat{h}_{\alpha_i} + \langle h_{\alpha_i}, \lambda - \rho \rangle + k \right)^{-1} \hat{e}_{-\alpha_i}^n \hat{e}_{\alpha_i}^n(T_i(v)), \\ \check{q}_{i,\lambda}(v) &= \sum_{n \geq 0} \frac{(-1)^n}{n!} \prod_{k=1}^n \left(-\hat{h}_{\alpha_i} + \langle h_{\alpha_i}, \lambda - \rho \rangle + k \right)^{-1} \hat{e}_{\alpha_i}^n \hat{e}_{-\alpha_i}^n(T_i(v)). \end{aligned}$$

More generally, for any element $w \in W$, the maps $\check{q}_w : \mathcal{A}' / \mathcal{A}' \mathbf{n} \rightarrow \mathcal{A}' / \mathcal{A}' \mathbf{n}$ and $\check{q}_w : {}_{\mathbf{n}_-}\mathcal{A}' \backslash \mathcal{A}' \rightarrow {}_{\mathbf{n}_-}\mathcal{A}' \backslash \mathcal{A}'$ define morphisms of modules over Mickelsson algebras

$$\check{q}_{w,\lambda} : {}_{\lambda, \mathbf{n}_-}\mathcal{A}_{\mathbf{n}} \rightarrow ({}_{w \circ \lambda, \mathbf{n}_-}\mathcal{A}_{\mathbf{n}})^{\check{q}_w} \quad \text{and} \quad \check{q}_{w,\lambda} : {}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n},\lambda} \rightarrow ({}_{\mathbf{n}_-}\mathcal{A}_{w \circ \lambda, \mathbf{n}})^{\check{q}_w}.$$

For a reduced decomposition $w = s_{\alpha_1} \cdots s_{\alpha_n}$ of an element $w \in W$ and a corresponding sequence of positive roots $\gamma_1, \dots, \gamma_n$, we have

$$\check{q}_{w,\lambda}(v) = \hat{p}_{\gamma_1}[\lambda - \rho] \cdots \hat{p}_{\gamma_n}[\lambda - \rho](T_w(v)), \quad \check{q}_{i,\lambda}(v) = \hat{p}_{-\gamma_1}[\lambda - \rho] \cdots \hat{p}_{-\gamma_n}[\lambda - \rho](T_w(v)).$$

Remark. By the definition, the double coset space ${}_{\mathbf{n}_-}\tilde{\mathcal{A}}_{\mathbf{n},\lambda} = {}_{\mathbf{n}_-}\mathcal{A} \backslash \mathcal{A} / \mathcal{A} \mathbf{n}$ is a left $S_{\mathbf{n}_-}(\mathcal{A})$ - and a right $S^n(\mathcal{A})$ -module. Its quotients

$$\begin{aligned} {}_{\mathbf{n}_-}\tilde{\mathcal{A}}_{\mathbf{n},\lambda} &= {}_{\mathbf{n}_-}\mathcal{A} \backslash \mathcal{A} / \mathcal{A} \cdot (\mathbf{n}, (h - \langle h, \lambda \rangle)|_{h \in \mathfrak{h}}), \\ {}_{\lambda, \mathbf{n}_-}\tilde{\mathcal{A}}_{\mathbf{n}} &= (\mathbf{n}_-, (h - \langle h, \lambda \rangle)|_{h \in \mathfrak{h}}) \mathcal{A} \backslash \mathcal{A} / \mathcal{A} \mathbf{n} \end{aligned}$$

coincide with the spaces (8.4) for generic λ : ${}_{\mathbf{n}_-}\tilde{\mathcal{A}}_{\mathbf{n},\lambda} = {}_{\mathbf{n}_-}\mathcal{A}_{\mathbf{n},\lambda}$ and ${}_{\lambda, \mathbf{n}_-}\tilde{\mathcal{A}}_{\mathbf{n}} = {}_{\lambda, \mathbf{n}_-}\mathcal{A}_{\mathbf{n}}$. They have correspondingly the structure of a left $S_{\mathbf{n}_-}(\mathcal{A})$ - and a right $S^n(\mathcal{A})$ -module.

The module ${}_{\mathbf{n}_-}\tilde{\mathcal{A}}_{\mathbf{n},\lambda}$ can be interpreted as a space of \mathbf{n}_- -coinvariants in the left \mathcal{A} -module $M_{\mathbf{n},\lambda}(\mathcal{A}) = \mathcal{A}/\mathcal{A} \cdot (\mathbf{n}, h - \langle h, \lambda \rangle | h \in \mathfrak{h})$. The module ${}_{\lambda, \mathbf{n}_-}\tilde{\mathcal{A}}_{\mathbf{n}}$ can be interpreted as a space of \mathbf{n} -coinvariants in the right \mathcal{A} -module $\tilde{M}_{\lambda}(\mathcal{A}) = (\mathbf{n}_-, h - \langle h, \lambda \rangle | h \in \mathfrak{h}) \mathcal{A} \backslash \mathcal{A}$.

For each i , the operators \check{q}_i and \check{q}_i define homomorphisms of Mickelsson algebras $S^n(\mathcal{A})$ and $S_{\mathbf{n}_-}(\mathcal{A})$ to their localizations with respect to denominators, generated by monomials $(h_{\alpha_i} + k)$, $k \in \mathbb{Z}$. If $\lambda \in \mathfrak{h}^*$ satisfies the condition $\langle h_{\alpha_i}, \lambda \rangle \notin \mathbb{Z}$, these localizations act in $\left({}_{s_{\alpha_i} \circ \lambda, \mathbf{n}_-}\tilde{\mathcal{A}}_{\mathbf{n}}\right)^{\check{q}_i}$ and $\left({}_{\mathbf{n}_-}\tilde{\mathcal{A}}_{\mathbf{n}, s_{\alpha_i} \circ \lambda}\right)^{\check{q}_i}$ correspondingly. In this sense the operators \check{q}_i and \check{q}_i define morphisms of the right $S^n(\mathcal{A})$ -modules and of the left $S_{\mathbf{n}_-}(\mathcal{A})$ -modules:

$$(8.5) \quad \check{q}_{i,\lambda} : {}_{\lambda, \mathbf{n}_-}\tilde{\mathcal{A}}_{\mathbf{n}} \rightarrow \left({}_{s_{\alpha_i} \circ \lambda, \mathbf{n}_-}\tilde{\mathcal{A}}_{\mathbf{n}}\right)^{\check{q}_i} \quad \text{and} \quad \check{q}_{i,\lambda} : {}_{\mathbf{n}_-}\tilde{\mathcal{A}}_{\mathbf{n},\lambda} \rightarrow \left({}_{\mathbf{n}_-}\tilde{\mathcal{A}}_{\mathbf{n}, s_{\alpha_i} \circ \lambda}\right)^{\check{q}_i}.$$

One can regard (8.5) as a family of operators with a meromorphic dependence on a parameter λ , study their singularities, residues etc.

8.2. Quotients of free modules. As any associative algebra with unit, the Mickelsson algebra is a free left and a free right module over itself of rank one. Let us restrict ourselves to the Mickelsson algebra $Z^n(\mathcal{A})$ and its free right module of rank one.

Let $\lambda \in \mathfrak{h}^*$ be a generic weight. Consider the following quotient of the free right $Z^n(\mathcal{A})$ -module

$$(8.6) \quad \Phi_{\lambda}(\mathcal{A}) = (h - \langle h, \lambda \rangle) |_{h \in \mathfrak{h}} Z^n(\mathcal{A}) \backslash Z^n(\mathcal{A}).$$

It can be realized as follows. The multiplication m in \mathcal{A} induces an isomorphism of the two left $U(\mathfrak{g})$ -modules:

$$(8.7) \quad \mathcal{V} \otimes M_{\mathbf{n}}(\mathfrak{g}) \rightsquigarrow M_{\mathbf{n}}(\mathcal{A}'),$$

where $M_{\mathbf{n}}(\mathfrak{g})$ is the 'universal Verma module' $U'(\mathfrak{g})/U'(\mathfrak{g})\mathbf{n}$, $M_{\mathbf{n}}(\mathcal{A}') = \mathcal{A}'/A'\mathbf{n}$, and \mathcal{V} is taken with a structure of the adjoint representation of \mathfrak{g} . The map (8.7) is also an isomorphism of the right D -modules, where the structure of D -modules in the left hand side of (8.7) is given by the prescription $(v \otimes m) \cdot d = v \otimes (m \cdot d)$ for any $v \in \mathcal{V}$, $m \in M_{\mathbf{n}}(\mathfrak{g})$, $d \in D$. The Mickelsson algebra is the space of highest weight vectors in $M_{\mathbf{n}}(\mathcal{A}')$, so with the identification (8.7) we have the following isomorphism of D -bimodules:

$$(8.8) \quad Z^n(\mathcal{A}) \approx (\mathcal{V} \otimes M_{\mathbf{n}}(\mathfrak{g}))^{\mathbf{n}}.$$

Recall that $Z^n(\mathcal{A})$ is a $U(\mathfrak{h})$ -bimodule and admits the weight decomposition with respect to the adjoint action of \mathfrak{h} . This implies that the right $Z^n(\mathcal{A})$ -module $\Phi_{\lambda}(\mathcal{A})$ is a semisimple right $U(\mathfrak{h})$ -module and admits a decomposition

$$\Phi_{\lambda}(\mathcal{A}) = \oplus_{\nu} \Phi_{\lambda, \lambda - \nu},$$

where the sum is taken over the weights ν of \mathcal{A} such that for any $\varphi \in \Phi_{\lambda, \mu}(\mathcal{A})$ we have

$$(8.9) \quad \varphi \cdot h = \langle h, \mu \rangle \varphi,$$

and $\Phi_{\lambda, \mu}$ coincides with the double coset

$$\Phi_{\lambda, \mu} = (h - \langle h, \lambda \rangle) |_{h \in \mathfrak{h}} \cdot Z^n(\mathcal{A}) \backslash Z^n(\mathcal{A}) / Z^n(\mathcal{A}) \cdot (h - \langle h, \mu \rangle) |_{h \in \mathfrak{h}}.$$

Analogously, the left $Z^n(\mathcal{A})$ -module $\Phi_{\cdot, \mu}(\mathcal{A})$

$$\Phi_{\cdot, \mu}(\mathcal{A}) = Z^n(\mathcal{A}) / Z^n(\mathcal{A}) \cdot (h - \langle h, \mu \rangle) |_{h \in \mathfrak{h}}$$

admits a presentation

$$\Phi_{\cdot,\mu}(\mathcal{A}) = \oplus_{\nu} \Phi_{\mu+\nu,\mu}.$$

With the identification (8.8), the relation (8.9) can be interpreted as follows: any element $\phi \in \Phi_{\lambda,\mu}(\mathcal{A})$ is a highest weight vector in the tensor product $\mathcal{V} \otimes M_{\mu}$ of the weight λ . Here $M_{\mu} = U(\mathfrak{g})/U(\mathfrak{g}) \cdot (\mathfrak{n}, (h - \langle h, \mu \rangle)h \in \mathfrak{h})$ is the Verma module of g with the highest weight μ . We have proved

Lemma 8.4. *Let $\lambda \in \mathfrak{h}^*$ be generic and $\nu \in \mathfrak{h}^*$ a weight of $Z^n(\mathcal{A})$. Then the weight space $\Phi_{\lambda,\lambda-\nu}$ of the right $Z^n(\mathcal{A})$ -module $\Phi_{\lambda}(\mathcal{A})$ is isomorphic to the space of intertwining operators*

$$(8.10) \quad \Phi_{\lambda,\lambda-\nu} \approx \text{Hom}_{U(\mathfrak{g})}(M_{\lambda}, \mathcal{V} \otimes M_{\lambda-\nu}).$$

Denote by \mathbb{I}_{λ} the class of unit $1 \in Z^n(\mathcal{A})$ in $\Phi_{\lambda}(\mathcal{A})$. The vector \mathbb{I}_{λ} generates $\Phi_{\lambda}(\mathcal{A})$ as a $Z^n(\mathcal{A})$ -module. For any $v \in \mathcal{V}$ denote by Φ_{λ}^v the vector of the right $Z^n(\mathcal{A})$ -module $\Phi_{\lambda}(\mathcal{A})$, obtained by an application of the element z'_v to \mathbb{I}_{λ} . It is equal to the class of z'_v in $\Phi_{\lambda}(\mathcal{A})$. Let ν be the weight of v . In the description (8.10), Φ_{λ}^v presents a map $\Phi_{\lambda}^v \in \text{Hom}_{U(\mathfrak{g})}(M_{\lambda}, \mathcal{V} \otimes M_{\lambda-\nu})$, such that

$$\Phi_{\lambda}^v(1_{\lambda}) = v \otimes 1_{\lambda-\nu} + \text{l.o.t.}$$

where 1_{λ} is the highest weight vector of Verma module M_{λ} and l.o.t. mean terms which have lower weight on the second tensor component.

The properties of the operators \check{q}_w imply that $\check{q}_w(\Phi_{\lambda,\mu}) = \Phi_{w \circ \lambda, w \circ \mu}$, so that each operator \check{q}_w defines morphisms of the right and left $Z^n(\mathcal{A})$ -modules:

$$(8.11) \quad \check{q}_{w,\lambda} : \Phi_{\lambda}(\mathcal{A}) \rightarrow (\Phi_{w \circ \lambda})^{\check{q}_w}(\mathcal{A}), \quad \check{q}_{w,\cdot,\mu} : \Phi_{\cdot,\mu}(\mathcal{A}) \rightarrow (\Phi_{\cdot,w \circ \mu})^{\check{q}_w}(\mathcal{A}).$$

Proposition 6.6 gives a formula for transformations of vectors Φ_{λ}^v :

$$(8.12) \quad \check{q}_{w,\lambda}(\Phi_{\lambda}^v) = \Phi_{w \circ \lambda}^{\hat{p}_{-\gamma_1}[\lambda+\rho] \cdots \hat{p}_{-\gamma_n}[\lambda+\rho](T_w(v))},$$

where $\gamma_1, \dots, \gamma_n$ is the sequence of positive roots, attached to a reduced decomposition $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_n}}$ by the standard rule $\gamma_1 = \alpha_{i_1}$, $\gamma_2 = s_{\alpha_{i_1}}(\alpha_{i_2})$, \dots .

8.3. Relations to dynamical Weyl group. Let \mathcal{V} be a $U(\mathfrak{g})$ -module algebra with a locally nilpotent action of real root vectors and $\mathcal{A}_{\mathcal{V}} = U(\mathfrak{g}) \ltimes \mathcal{V}$ be a smash product of $U(\mathfrak{g})$ and \mathcal{V} , see the example 2 in Section 3.1.

The typical examples are: the tensor algebra of an integrable highest weight representation of \mathfrak{g} and the symmetric algebra of an integrable highest weight representation of \mathfrak{g} .

Due to assumptions above, the \mathfrak{g} -module structure in \mathcal{V} lifts to an action of the Weyl group of \mathfrak{g} in \mathcal{V} by standard formulas

$$(8.13) \quad \hat{T}_i = \exp \hat{e}_{\alpha_i} \cdot \exp -\hat{e}_{-\alpha_i} \cdot \exp \hat{e}_{\alpha_i}.$$

Due to (8.13), operators \hat{T}_i are automorphisms of the algebra \mathcal{V} .

The operators \hat{T}_i admit a lift to automorphisms of \mathcal{A} by the relation $T_i(gv) = T_i(g)\hat{T}_i(v)$, where $g \in U(\mathfrak{g})$, $v \in \mathcal{V}$, and $T_i(g)$ is the automorphism of $U(\mathfrak{g})$, as in Section 2.1. Actually, they are given by the same relation (8.13) with respect to the adjoint action of \mathfrak{g} in \mathcal{A} .

Elements of right $Z^n(\mathcal{A})$ -modules $\Phi_{\lambda}(\mathcal{A})$ are known in this case under the name *intertwining operators*. The morphisms $\check{q}_{w,\lambda}$ generate a so called *dynamical Weyl group* action, see [EV, TV].

The intertwining operators $\Phi_\lambda(\mathcal{A})$ form an algebroid with respect to the composition operation. The composition of intertwining operators can be described as follows.

For a generic $\lambda \in \mathfrak{h}^*$, any morphism $\varphi_\lambda : M_\lambda \rightarrow \mathcal{V} \otimes M_{\lambda-\nu}$ of \mathfrak{g} -modules admits a lift to a morphism $\bar{\varphi}_\lambda : \mathcal{V} \otimes M_\lambda \rightarrow \mathcal{V} \otimes M_{\lambda-\nu}$ of \mathcal{A}' -modules by the rule $\bar{\varphi}_\lambda(v \otimes m) = v \cdot \varphi_\lambda(m)$ for any $m \in M_\lambda$. Then the composition $\varphi'_{\lambda-\nu} \circ \varphi_\lambda$ of intertwining operators $\varphi_\lambda \in \Phi_{\lambda, \lambda-\nu}$ and $\varphi'_{\lambda-\nu} \in \Phi_{\lambda-\nu, \lambda-\nu-\nu'}$ is an element $\varphi''_\lambda \in \Phi_{\lambda, \lambda-\nu-\nu'}$, such that

$$\bar{\varphi}''_\lambda = \bar{\varphi}'_{\lambda-\nu} \circ \bar{\varphi}_\lambda.$$

The composition of intertwining operators coincides with the structure of the right $Z^n(\mathcal{A})$ -module in Φ_λ . Namely, in the notation of the previous section, for any $x \in Z^n(\mathcal{A})$, denote by Φ_λ^x its class in Φ_λ , considered as intertwining operator. Then we have

Proposition 8.5. [K] *Let $z', z'' \in Z^n(\mathcal{A})$. Assume that the weight of z' with respect to the adjoint action of \mathfrak{h} is ν . Then*

$$\Phi_\lambda^{z'z''} = \Phi_{\lambda-\nu}^{z''} \circ \Phi_\lambda^{z'}.$$

The statement that the maps $\check{q}_{w,\lambda}$ are morphisms of $Z^n(\mathcal{A})$ -modules, see (8.11), is equivalent in this context to the statement that the dynamical Weyl group action respects the composition of intertwining operators.

9. QUANTUM GROUP SETTINGS

9.1. Notation and assumptions. In this section we announce basic statements of the paper for Mickelsson algebras, related to reductions over quantum groups. We restrict our attention to the Mickelsson algebras $Z^n(\mathcal{A})$.

Keep the notation of Section 2.1. Let ν be an indeterminate; $d_i \in \mathbb{N}$ are defined by the condition that the matrix $(\alpha_i, \alpha_j) = d_i a_{i,j} = d_j a_{j,i}$ is symmetric. Here $a_{i,j}$ is the Cartan matrix of \mathfrak{g} . For any root $\gamma \in \Delta$ put $\nu_\gamma = \nu^{(\gamma, \gamma)/2}$, $[a]_p = \frac{p^a - p^{-a}}{p - p^{-1}}$, and $(a)_p = \frac{p^a - 1}{p - 1}$ for any symbols a and p . We also use the notation $\nu_i = \nu_{\alpha_i} = \nu^{d_i}$ for simple roots α_i .

Denote by $U_\nu(\mathfrak{g})$ the Hopf algebra, generated by Chevalley generators $e_{\alpha_i} \in U_\nu(\mathfrak{n})$, $e_{-\alpha_i} = f_{\alpha_i} \in U_\nu(\mathfrak{n}_-)$, $k_{\alpha_i}^{\pm 1} = \nu_i^{\pm h_{\alpha_i}} \in U_\nu(\mathfrak{h})$, where $\alpha_i \in \Pi$, so that

$$\begin{aligned} [e_{\alpha_i}, e_{-\alpha_j}] &= \delta_{i,j} [h_{\alpha_i}]_{q_i} = \delta_{i,j} \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{\nu_i - \nu_i^{-1}}, \\ k_{\alpha_i} e_{\pm \alpha_j} k_{\alpha_i}^{-1} &= \nu_i^{\pm a_{i,j}} e_{\pm \alpha_j} = \nu^{\pm(\alpha_i, \alpha_j)} e_{\pm \alpha_j}, \\ \sum_{r+s=1-a_{i,j}} (-1)^r e_{\pm \alpha_i}^{(r)} e_{\pm \alpha_j} e_{\pm \alpha_i}^{(s)} &= 0, \quad i \neq j, \quad \text{where } e_{\pm \alpha_i}^{(k)} = \frac{e_{\pm \alpha_i}^k}{[k]_{\nu_i}!}, \\ \Delta(e_{\alpha_i}) &= e_{\alpha_i} \otimes 1 + k_{\alpha_i} \otimes e_{\alpha_i}, & \Delta(e_{-\alpha_i}) &= 1 \otimes e_{-\alpha_i} + e_{-\alpha_i} \otimes k_{\alpha_i}^{-1}, \\ \Delta(k_{\alpha_i}) &= k_{\alpha_i} \otimes k_{\alpha_i}, & S(k_{\alpha_i}) &= k_{\alpha_i}^{-1}, \\ S(e_{\alpha_i}) &= -k_{\alpha_i}^{-1} e_{\alpha_i}, & S(e_{-\alpha_i}) &= -e_{-\alpha_i} k_{\alpha_i}. \end{aligned}$$

The adjoint action (3.1) of the Chevalley generators has the form

$$\begin{aligned} \hat{e}_{\alpha_i}(x) &\equiv \text{ad}_{e_{\alpha_i}}(x) = e_{\alpha_i} x - k_{\alpha_i} x k_{\alpha_i}^{-1} e_{\alpha_i}, \\ \hat{e}_{-\alpha_i}(x) &\equiv \text{ad}_{e_{-\alpha_i}}(x) = [e_{-\alpha_i}, x] \cdot k_{\alpha_i}. \end{aligned} \tag{9.1}$$

Let $T_i \equiv T_{s_i} : U_\nu(\mathfrak{g}) \rightarrow U_\nu(\mathfrak{g})$ be automorphisms of $U_\nu(\mathfrak{g})$, defined by the relations

$$(9.2) \quad \begin{aligned} T_i(e_{\alpha_i}) &= -k_{\alpha_i} e_{-\alpha_i}, & T_i(e_{-\alpha_i}) &= -e_{\alpha_i} k_{\alpha_i}^{-1}, & T_i(k_{\alpha_j}) &= k_{s_{\alpha_i}(\alpha_j)}, \\ T_i(e_{\alpha_j}) &= \sum_{r+s=1-a_{i,j}} (-1)^r \nu_i^r e_{\alpha_i}^{(r)} e_{\alpha_j} e_{\alpha_i}^{(s)}, & i &\neq j, \\ T_i(e_{-\alpha_j}) &= \sum_{r+s=1-a_{i,j}} (-1)^r \nu_i^{-r} e_{-\alpha_i}^{(s)} e_{-\alpha_j} e_{-\alpha_i}^{(r)}, & i &\neq j. \end{aligned}$$

Here we use the Cartan generators k_γ , $\gamma \in \Delta$. They are defined by the rules $k_{-\alpha} = k_\alpha^{-1}$, and $k_{\alpha+\beta} = k_\alpha k_\beta$.

In Lusztig's notation [L], $T_i \equiv T'_{i,+}$. For any $w \in W$ they define automorphisms $T_w : U_\nu(\mathfrak{g}) \rightarrow U_\nu(\mathfrak{g})$, as in Section 6.1.

Denote by D_ν the localization of the commutative algebra $U_\nu(\mathfrak{h})$ with respect to the multiplicative set of denominators, generated by

$$\{[h_\alpha + k]_{\nu_\alpha} | \alpha \in \Delta, k \in \mathbb{Z}\}.$$

Denote by $U'_\nu(\mathfrak{g})$ the extension of $U_\nu(\mathfrak{g})$ by means of D_ν :

$$U'_\nu(\mathfrak{g}) = U_\nu(\mathfrak{g}) \otimes_{U_\nu(\mathfrak{h})} D_\nu \approx D_\nu \otimes_{U_\nu(\mathfrak{h})} U_\nu(\mathfrak{g}).$$

As well as in the nondeformed case ($\nu = 1$), there exists an extension $F_{\mathfrak{g},n}^\nu$ of the algebra $U'_\nu(\mathfrak{g})$ and an element $P = P_n \in F_{\mathfrak{g},n}^\nu$ (the extremal projector), satisfying the conditions $e_{\alpha_i} P = P e_{-\alpha_i} = 0$, $P^2 = P$, see [KT].

In particular, for the algebra $U_q(\mathfrak{sl}_2)$, generated by $e_{\pm\alpha}$ and $k_\alpha^{\pm 1}$, we have two projection operators, $P = P_\alpha[\rho]$, and $P_- = P_{-\alpha}[\rho]$, where

$$(9.3) \quad \begin{aligned} P_\alpha[\lambda] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{[n]_{\nu_\alpha}!} \nu_\alpha^{-\langle h_\alpha, \lambda - \rho \rangle} f_{\alpha,n}[\lambda] e_{-\alpha}^n e_\alpha^n, \\ P_{-\alpha}[\lambda] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{[n]_{\nu_\alpha}!} \nu_\alpha^{\langle h_\alpha, \lambda - \rho \rangle} g_{\alpha,n}[\lambda] e_\alpha^n e_{-\alpha}^n \end{aligned}$$

and

$$f_{\alpha,n}[\lambda] = \prod_{j=1}^n [h_\alpha + \langle h_\alpha, \lambda \rangle + j]_{\nu_\alpha}^{-1}, \quad g_{\alpha,n}[\lambda] = \prod_{j=1}^n [-h_\alpha + \langle h_\alpha, \lambda \rangle + j]_{\nu_\alpha}^{-1}.$$

Let \mathcal{A} be an associative algebra, which contains $U_\nu(\mathfrak{g})$. We call \mathcal{A} a $U_\nu(\mathfrak{g})$ -admissible algebra if

- (a) there is a subspace $\mathcal{V} \subset \mathcal{A}$, invariant with respect to the adjoint action of $U_\nu(\mathfrak{g})$, such that the multiplication m in \mathcal{A} induces an isomorphisms of vector spaces

$$(a1) \quad m : U_\nu(\mathfrak{g}) \otimes \mathcal{V} \rightarrow \mathcal{A}, \quad (a2) \quad m : \mathcal{V} \otimes U_\nu(\mathfrak{g}) \rightarrow \mathcal{A};$$

- (b) the adjoint action in \mathcal{V} of all real root vectors $e_\gamma \in U_\nu(\mathfrak{g})$, related to any normal ordering of the root system, is locally nilpotent. The adjoint action of the Cartan subalgebra $U_\nu(\mathfrak{h})$ in \mathcal{V} is semisimple.

In particular, \mathcal{A} is isomorphic to $U_\nu(\mathfrak{g}) \otimes \mathcal{V}$ and to $\mathcal{V} \otimes U_\nu(\mathfrak{g})$ as a $U_\nu(\mathfrak{g})$ -module with respect to the adjoint action.

Denote by \mathfrak{n} the linear subspace of $U_\nu(\mathfrak{g})$ generated by the elements e_{α_i} , $\alpha_i \in \Pi$. Denote by \mathfrak{n}_- the linear subspace of $U_\nu(\mathfrak{g})$ generated by the elements $e_{-\alpha_i}$, $i \in \Pi$. Let

$U_\nu(\mathfrak{n})$ be the subalgebra of $U_\nu(\mathfrak{g})$, generated by \mathfrak{n} and $U_\nu(\mathfrak{n}_-)$ the subalgebra of $U_\nu(\mathfrak{g})$, generated by \mathfrak{n}_- .

For any $U_\nu(\mathfrak{g})$ -admissible algebra \mathcal{A} put $\mathcal{A}' = \mathcal{A} \otimes_{U_\nu(\mathfrak{h})} D_\nu$ and define Mickelsson algebras $S^n(\mathcal{A})$ and $Z^n(\mathcal{A})$, as in Section 3.2:

$$S^n(\mathcal{A}) = \text{Nr}(\mathcal{A}\mathfrak{n})/\mathcal{A}\mathfrak{n}, \quad Z^n(\mathcal{A}) = \text{Nr}(\mathcal{A}'\mathfrak{n})/\mathcal{A}'\mathfrak{n}$$

and the double coset algebra ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$, see Section 3.3,

$${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} = \mathfrak{n}_-\mathcal{A}' \backslash \mathcal{A}' / \mathcal{A}'\mathfrak{n} \equiv \mathcal{A}' / (\mathfrak{n}_-\mathcal{A}' + \mathcal{A}'\mathfrak{n}),$$

equipped with the multiplication structure (3.7).

With the conditions of Section 3.4, the Mickelsson algebra $Z^n(\mathcal{A})$ has distinguished generators $z_v, z'_v, v \in \mathcal{V}$, determined by the relations (3.9) and (3.10).

9.2. Basic constructions. Let $\alpha \in \Pi$ be a simple root, $\mathfrak{n}_\alpha = \mathbb{C}e_\alpha$, $\mathfrak{n}_{-\alpha} = \mathbb{C}e_{-\alpha}$. Let $x \in \mathcal{A}$ be an element of \mathcal{A} , finite with respect to the adjoint action of e_α . Denote by $q_\alpha(x)$ the following element of $\mathcal{A}'/\mathcal{A}'\mathfrak{n}_\alpha$:

$$(9.4) \quad \begin{aligned} q_\alpha(x) &= \sum_{n \geq 0} \frac{(-1)^n}{[n]_{\nu_\alpha}!} \left(\hat{k}_\alpha^{-1} \hat{e}_\alpha \right)^n (x) e_{-\alpha}^n g_{n,\alpha} \\ &= \sum_{n \geq 0} \frac{(-1)^n}{[n]_{\nu_\alpha}!} \nu_\alpha^{-n(\hat{h}_\alpha - n + 1)} \hat{e}_\alpha^n (x) e_{-\alpha}^n g_{n,\alpha} \quad \text{mod } \mathcal{A}'\mathfrak{n}_\alpha, \end{aligned}$$

where $g_{n,\alpha} = ([h_\alpha]_{\nu_\alpha} [h_\alpha - 1]_{\nu_\alpha} \cdots [h_\alpha - n + 1]_{\nu_\alpha})^{-1}$. The assignment (9.4) has the properties

- (i) $q_\alpha(xe_{-\alpha}) = 0$,
- (ii) $q_\alpha(xk_\alpha^{-1}) = \nu_\alpha^{-2} q_\alpha(x) k_\alpha^{-1}$, $q_\alpha(k_\alpha^{-1}x) = \nu_\alpha^{-2} k_\alpha^{-1} q_\alpha(x)$,
- (iii) $q_\alpha(xk_\gamma) = q_\alpha(x) k_\gamma$, $q_\alpha(k_\gamma x) = k_\gamma q_\alpha(x)$, if $\langle h_\alpha, \gamma \rangle = 0$,
- (iv) $k_\alpha^{-1} e_\alpha q_\alpha(x) = q_\alpha(k_\alpha^{-1} e_\alpha x) = 0$.

We extend the assignment (9.4) to the map $q_\alpha : \mathcal{A}' \rightarrow \mathcal{A}'/\mathcal{A}'\mathfrak{n}_\alpha$ with the help of the properties (ii) and (iii). It satisfies the property

$$(9.5) \quad q_\alpha(xd) = q_\alpha(x) \tau_\alpha(d), \quad q_\alpha(dx) = \tau_\alpha(d) q_\alpha(x), \quad d \in D_\nu,$$

where $\tau_\mu : D_\nu \rightarrow D_\nu$, $\mu \in \mathfrak{h}^*$ is uniquely characterized by the conditions

$$\tau_\alpha(k_\gamma) = \nu^{(\mu, \gamma)} k_\gamma.$$

Due to the property (iv) the map q_α defines a map $q_\alpha : {}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}} \rightarrow Z^{\mathfrak{n}_\alpha}(\mathcal{A})$. We have an analog of Theorem 3:

Proposition 9.1. *The map*

$$q_\alpha : {}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}} \rightarrow Z^{\mathfrak{n}_\alpha}(\mathcal{A})$$

is an isomorphism of algebras.

In the following we assume that the automorphisms (9.2) admit extensions $T_i : \mathcal{A}' \rightarrow \mathcal{A}'$, which satisfy the braid group relations (6.1), though, as well as in the classical case, part of the results below do not depend on this assumption.

Let $w \in W$ be an element of the Weyl group of \mathfrak{g} , $\alpha \in \Pi$ a simple root, such that $l(ws_\alpha) = l(w) + 1$. Set $\gamma = w(\alpha)$, $e_{\pm\gamma} = T_w(e_{\pm\alpha})$ and $T_\gamma = T_w T_\alpha T_w^{-1}$. Denote by

$\mathfrak{m} = \mathfrak{n}^w$ the linear span of the vectors $T_w(e_{\alpha_i})$, $\alpha_i \in \Pi$ and by \mathfrak{m}^{s_γ} the space $T_\gamma(\mathfrak{m})$. Let $q_{\gamma, \mathfrak{m}}$ be the linear map $q_{\gamma, \mathfrak{m}} : \mathcal{A}' \rightarrow \mathcal{A}'/\mathcal{A}'\mathfrak{m}$, defined by the prescription:

$$\begin{aligned} q_{\gamma, \mathfrak{m}}(x) &= \sum_{n \geq 0} \frac{(-1)^n}{[n]_{\nu_\gamma}!} \left(\hat{k}_\gamma^{-1} \hat{e}_\gamma \right)^n (x) e_{-\gamma}^n g_{n, \gamma} \\ &= \sum_{n \geq 0} \frac{(-1)^n}{[n]_{\nu_\gamma}!} \nu_\gamma^{-n(\hat{h}_\gamma - n + 1)} \hat{e}_\gamma^n (x) e_{-\gamma}^n g_{n, \gamma} \quad \text{mod } \mathcal{A}'\mathfrak{m}, \end{aligned}$$

for x , which are adjoint finite with respect to e_γ , and then extended to \mathcal{A}' by means of the properties, analogous to (i) and (ii) for the map q_α .

Here $g_{n, \gamma} = ([h_\gamma]_{\nu_\gamma} [h_\gamma - 1]_{\nu_\gamma} \cdots [h_\gamma - n + 1]_{\nu_\gamma})^{-1}$.

Proposition 9.2.

- (i) We have $q_{\gamma, \mathfrak{m}}(\mathcal{A}'\mathfrak{m}^{s_\gamma}) = 0$, so $q_{\gamma, \mathfrak{m}}$ defines a map $q_{\gamma, \mathfrak{m}} : \mathcal{A}'/\mathcal{A}'\mathfrak{m}^{s_\gamma} \rightarrow \mathcal{A}'/\mathcal{A}'\mathfrak{m}$.
- (ii) We have an equality ($\alpha = w^{-1}(\gamma)$)

$$(9.6) \quad q_{\gamma, \mathfrak{m}} = T_w q_{\alpha, \mathfrak{n}} T_w^{-1}.$$

Note that the statement (ii) is nontrivial for $\nu \neq 1$, since

$$T_w \text{ ad }_x T_w^{-1}(y) \neq \text{ ad }_{T_w(x)}(y).$$

Let $\bar{w} = \{w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_n}}\}$ be a reduced decomposition of the element $w \in W$. Let $\gamma_1, \dots, \gamma_n$ be a related sequence of positive roots: $\gamma_1 = \alpha_{i_1}, \dots, \gamma_k = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_{k-1}}}(\alpha_{i_k}), \dots$. Proposition 9.2 (i) implies that there is a well defined map

$$q_{\bar{w}} : \mathcal{A}'/\mathcal{A}'\mathfrak{n}^w \rightarrow \mathcal{A}'/\mathcal{A}'\mathfrak{n} : \quad q_{\bar{w}} = q_{\gamma_1, \mathfrak{n}} q_{\gamma_2, \mathfrak{n}^{s_{\gamma_1}}} \cdots q_{\gamma_n, \mathfrak{n}^{s_{\gamma_1} \cdots s_{\gamma_{n-1}}}}.$$

Proposition 9.3. *Let \mathfrak{g} be of finite dimension. Then for any reduced decomposition \bar{w}_0 of the longest element w_0 of W the map $q_{\bar{w}_0}$ sends a vector $v \in \mathcal{V}$ to the generator z'_v of the Mickelsson algebra $Z^n(\mathcal{A})$.*

Proposition 9.3 implies that the maps $q_{\gamma, \mathfrak{m}}$ satisfy the cocycle conditions, that is, the maps $q_{\bar{w}}$ do not depend on a reduced decomposition \bar{w} of $w \in W$; they can thus be denoted as q_w .

Set $\check{q}_i = q_{s_{\alpha_i}} \cdot T_i : \mathcal{A}'/\mathcal{A}'\mathfrak{n} \rightarrow \mathcal{A}'/\mathcal{A}'\mathfrak{n}$. They satisfy the braid group relations:

$$\underbrace{\check{q}_i \check{q}_j \cdots}_{m_{i,j}} = \underbrace{\check{q}_j \check{q}_i \cdots}_{m_{i,j}}, \quad i \neq j,$$

and, due to Proposition 9.1, their restriction to $Z^n(\mathcal{A})$ are automorphisms of the Mickelsson algebra, such that

$$\check{q}_i(dx) = (s_{\alpha_i} \circ d) \cdot x, \quad \check{q}_i(xd) = x \cdot (s_{\alpha_i} \circ d), \quad d \in D_\nu, x \in \mathcal{A}'.$$

Here $w \circ d$ is the natural extension of the shifted action of $w \in W$ in \mathfrak{h}^* , see (6.8), to the automorphism of D_ν , defined by the conditions

$$w \circ k_\gamma = k_{w(\gamma)} \cdot q_\gamma^{\langle \gamma, w(\rho - \rho) \rangle}.$$

9.3. Some calculations. There is a standard construction of the extension of the Hopf algebra $U_\nu(\mathfrak{g})$ by means of automorphisms T_i . Namely, let $U_\nu^W(\mathfrak{g})$ be the smash product of $U_\nu(\mathfrak{g})$ and of the algebra, generated by elements $T_i^{\pm 1}$, satisfying the braid group relations

$$\underbrace{T_i T_j \cdots}_{m_{i,j}} = \underbrace{T_j T_i \cdots}_{m_{i,j}}, \quad i \neq j.$$

The cross-product relations are

$$(9.7) \quad T_i g T_i^{-1} = T_i(g), \quad g \in U_\nu(\mathfrak{g}).$$

Due to coalgebraic properties of Lusztig automorphism, the smash product $U_\nu^W(\mathfrak{g})$ can be equipped with a structure of a Hopf algebra, if we put

$$\Delta(T_i) = T_i \otimes T_i \cdot \tilde{R}_i,$$

where

$$\tilde{R}_i = \exp_{\nu_i^{-2}}((\nu_i - \nu_i^{-1})e_{-\alpha_i} \otimes e_{\alpha_i}) = \sum_{n \geq 0} \frac{(\nu_i - \nu_i^{-1})^n}{(n)_{\nu_i^{-2}}!} e_{-\alpha_i}^n \otimes e_{\alpha_i}^n.$$

In the same way we extend the algebra \mathcal{A}' to the cross-product \mathcal{A}^W , using the relations (9.7). Since U_ν^W is a Hopf algebra, the adjoint action \hat{T}_i of T_i in \mathcal{A}^W is well defined. It respects the subalgebra $\mathcal{A}' \subset \mathcal{A}^W$: $\hat{T}_i(\mathcal{A}') \subset \mathcal{A}'$. The following statement is nontrivial for $\nu \neq 1$ and important for calculations of the maps \check{q}_i .

Proposition 9.4. *For any $x \in \mathcal{A}'$ we have*

$$\check{q}_i(x) = q_i(T_i x T_i^{-1}) = q_i(\hat{T}_i(x)).$$

Now we describe the squares of the automorphisms $\check{q}_i : Z^n(\mathcal{A}) \rightarrow Z^n(\mathcal{A})$. Assume that elements $v_{m,j} \in \mathcal{V}$, $m \in \mathbb{Z}_{\geq 0}$, $j = 0, 1, \dots, m$ form a finite-dimensional representation of the algebra $U_\nu(\mathfrak{sl}_2)$, generated by $e_{\pm \alpha_i}$ and $k_{\alpha_i}^{\pm 1}$, with respect to the adjoint action, so that we have:

$$(9.8) \quad \hat{e}_{\alpha_i}^{j+1}(v_{m,j}) = \hat{e}_{-\alpha_i}^{m-j+1}(v_{m,j}) = 0, \quad \hat{h}_{\alpha_i}(v_{m,j}) = (m - 2j)v_{m,j}.$$

In particular, $v_m = v_{m,0}$ is the highest weight vector of this representation, and $v_{m,j} = \hat{e}_{-\alpha_i}^{(j)}(v_m)$.

Proposition 9.5. *Assume that $v_{m,j} \in \mathcal{V}$ satisfy (9.8). Then*

$$(9.9) \quad \check{q}_i^2(v_{m,j}) = \nu^{-j(m-j+1)-(j+1)(m-j)} \cdot [h_{\alpha_i} + 1]_{\nu_i}^{-1} \cdot \hat{T}_i^2(v_{m,j}) \cdot [h_{\alpha_i} + 1]_{\nu_i}.$$

The property (9.9) simplifies under natural assumptions on operators $\hat{T}_i : \mathcal{V} \rightarrow \mathcal{V}$. Namely, suppose that the operators $\hat{T}_i : \mathcal{V} \rightarrow \mathcal{V}$ satisfy properties of Lusztig symmetries $T'_{i,+}$, that is, (see [L], 5.2.2)

$$(9.10) \quad \hat{T}_i(v_{m,j}) = \nu^{j(m+1-j)} v_{m,m-j}.$$

Corollary 9.6. *With the conditions (9.10) for any $x \in Z^n(A)$ we have*

$$(9.11) \quad \check{q}_i^2(x) = [h_{\alpha_i} + 1]_{\nu_i}^{-1} \cdot x \cdot [h_{\alpha_i} + 1]_{\nu_i}.$$

Keep the notation (3.12) of Section 3.4. For a real root $\alpha \in \Delta^{re}$, set

$$\begin{aligned}\bar{f}_{n,\alpha}^{(2)}[\mu] &= \nu_\alpha^{-2n} (k_\alpha^{(1)})^{-n} \prod_{k=1}^n \left([\hat{h}_\alpha^{(2)} - h_\alpha^{(1)} + \langle h_\alpha, \mu \rangle + k]_{\nu_\alpha} \right)^{-1}, \\ \bar{g}_{n,\alpha}^{(2)}[\mu] &= (k_\alpha^{(1)})^n \prod_{k=1}^n \left([-\hat{h}_\alpha^{(2)} + h_\alpha^{(1)} + \langle h_\alpha, \mu \rangle + k]_{\nu_\alpha} \right)^{-1}.\end{aligned}$$

Here $\hat{h}_\alpha^{(2)} = \text{ad}_{h_\alpha}^{(2)}$ is the adjoint action of h_α in \mathcal{V} , $h_\alpha^{(1)}$ is the operator of multiplication by h_α in D . For $\mu \in \mathfrak{h}^*$ define operators $C_\alpha^{(2)}[\mu] : D \otimes \mathcal{V} \rightarrow D \otimes \mathcal{V}$ and $B_{-\alpha}^{(2)}[\mu] : D \otimes \mathcal{V} \rightarrow D \otimes \mathcal{V}$ by the relations (3.16):

$$\begin{aligned}C_\alpha^{(2)}[\mu] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{[n]_{\nu_\alpha}!} \bar{f}_{n,\alpha}^{(2)}[\mu] (\hat{e}_{-\alpha}^{(2)})^n (\hat{e}_\alpha^{(2)})^n, \\ B_{-\alpha}^{(2)}[\mu] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{[n]_{\nu_\alpha}!} \bar{g}_{n,\alpha}^{(2)}[\mu] (\hat{e}_\alpha^{(2)})^n (\hat{e}_{-\alpha}^{(2)})^n.\end{aligned}$$

Proposition 9.7. *For any $v \in \mathcal{V}$ we have*

$$\check{q}_i(z'_v) = z'_{B_{-\alpha_i}^{(2)}[\rho](1 \otimes T_i(v))}, \quad \check{q}_i(z_v) = z_{C_{\alpha_i}^{(2)}[-\rho](1 \otimes T_i(v))}.$$

9.4. Another adjoint action. In this Section we sketch the modification of the above constructions for the second adjoint action.

Let $U_\nu^{op}(\mathfrak{g})$ be the Hopf algebra $U_\nu(\mathfrak{g})$, described in Section 9.1, with the same multiplication and opposite comultiplication,

$$\Delta^{op}(e_{\alpha_i}) = k_{\alpha_i} \otimes e_{\alpha_i} + e_{\alpha_i} \otimes 1, \quad \Delta^{op}(e_{-\alpha_i}) = e_{-\alpha_i} \otimes k_{\alpha_i}^{-1} + 1 \otimes e_{-\alpha_i}.$$

The adjoint action (3.1) for the algebra $U_\nu^{op}(\mathfrak{g})$ looks slightly different:

$$(9.12) \quad \begin{aligned}\hat{e}_{\alpha_i}(x) &\equiv \text{ad}_{e_{\alpha_i}}(x) = [e_{\alpha_i}, x] \cdot k_{\alpha_i}^{-1}, \\ \hat{e}_{-\alpha_i}(x) &\equiv \text{ad}_{e_{-\alpha_i}}(x) = e_{-\alpha_i}x - k_{\alpha_i}^{-1}xk_{\alpha_i}e_{-\alpha_i}.\end{aligned}$$

Let \mathcal{A} be a $U_\nu^{op}(\mathfrak{g})$ -admissible algebra. We define Zhelobenko operators, starting from the assignment

$$\begin{aligned}q_\alpha(x) &= \sum_{n \geq 0} \frac{(-1)^n}{[n]_{\nu_\alpha}!} (\hat{e}_\alpha)^n(x) (e_{-\alpha}^n k_\alpha) g_{n,\alpha} \\ &= \sum_{n \geq 0} \frac{(-1)^n}{[n]_{\nu_\alpha}!} \hat{e}_\alpha^n(x) e_{-\alpha}^n \nu_\alpha^{n(h_\alpha - n + 1)} g_{n,\alpha} \quad \text{mod } \mathcal{A}'\mathfrak{n}_\alpha.\end{aligned}$$

Corresponding maps $q_{\alpha,\mathfrak{m}}$ satisfy the cocycle conditions, such that the operators

$$\check{q}_i = q_{\alpha_i,\mathfrak{n}} \cdot T_i^{-1}$$

are automorphisms of the Mickelsson algebra $Z^n(\mathcal{A})$, satisfying the braid group relations.

9.5. Relations to dynamical Weyl group. Let \mathcal{V} be a $U_\nu(\mathfrak{g})$ -module algebra with a locally nilpotent action of real root vectors of $U_\nu(\mathfrak{g})$. It means that \mathcal{V} is an associative algebra and a $U_\nu(\mathfrak{g})$ -module, such that for any $g \in U_\nu(\mathfrak{g})$, $v_1, v_2 \in \mathcal{V}$, we have the equality

$$\hat{g}(v_1 \cdot v_2) = \sum_i \hat{g}'_i(v_1) \cdot \hat{g}''_i(v_2).$$

Here $\Delta(g) = \sum_i g'_i \otimes g''_i$, and $\hat{g}(v)$ is an action of $g \in U_\nu(\mathfrak{g})$ on $v \in \mathcal{V}$. Suppose that \mathcal{V} is equipped with an action of operators \hat{T}_i , which are automorphisms of the algebra, satisfy the braid group relations, and form an equivariant structure with respect to the operators (9.2), that is, for any $v \in \mathcal{V}$, $g \in U_\nu(\mathfrak{g})$,

$$\hat{T}_i(\hat{g}(v)) = T_i(g)\hat{T}_i(v),$$

where $T_i(g)$ means the application of operators (9.2).

These conditions imply that \mathcal{V} is a module algebra over $U_\nu^W(\mathfrak{g})$. So we have a well defined smash product $U_\nu^W(\mathfrak{g}) \ltimes \mathcal{V}$, which contains $U_\nu(\mathfrak{g}) \ltimes \mathcal{V}$ and the elements T_i . The automorphisms $T_i: U_\nu^W(\mathfrak{g}) \ltimes \mathcal{V} \rightarrow U_\nu^W(\mathfrak{g}) \ltimes \mathcal{V}$, given as $T_i(x) = T_i x T_i^{-1}$, preserve the subalgebra $U_\nu(\mathfrak{g}) \ltimes \mathcal{V}$. Moreover, the restriction of the adjoint action of T_i on \mathcal{V} coincides with \hat{T}_i : $\hat{T}_i|_{\mathcal{V}} = \hat{T}_i$. Thus the smash product $\mathcal{A} = U_\nu(\mathfrak{g}) \ltimes \mathcal{V}$ is a $U_\nu(\mathfrak{g})$ -admissible algebra.

Elements of the right $Z^n(\mathcal{A})$ -module $\Phi_\lambda(\mathcal{A})$, defined in (8.6), are intertwining operators $\Phi_\lambda^v: M_\lambda \rightarrow \mathcal{V} \otimes M_\lambda$. Operators \check{q}_i give rise to the operators $\check{q}_{i,\lambda}$ of the dynamical Weyl group

$$\check{q}_{i,\lambda}(\Phi_\lambda^v) = \Phi_{s_{\alpha_i}^{-1}[\lambda+\rho](\hat{T}_i(v))}^{\check{p}_{-\alpha_i}[\lambda+\rho](\hat{T}_i(v))}$$

and

$$\check{q}_{w,\lambda}(\Phi_\lambda^v) = \Phi_{w \circ \lambda}^{\hat{p}_{-\gamma_1}[\lambda+\rho] \cdots \hat{p}_{-\gamma_n}[\lambda+\rho](\hat{T}_w(v))},$$

where $\gamma_1, \dots, \gamma_n$ is the sequence of positive roots, attached to a reduced decomposition $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_n}}$ by the standard rule $\gamma_1 = \alpha_{i_1}$, $\gamma_2 = s_{\alpha_{i_1}}(\alpha_{i_2}), \dots$; here $\check{p}_{-\gamma_k}[\lambda+\rho]$ is the adjoint action of the operators (9.3), $e_{\pm\gamma_k} = T_{\alpha_{i_1}} \cdots T_{\alpha_{i_{k-1}}}(e_{\pm\alpha_{i_k}})$ are the Cartan-Weyl generators.

10. CONCLUDING REMARKS

We conclude with remarks on the assumptions on a \mathfrak{g} -admissible algebra \mathcal{A} , used in the paper. They are listed in Section 3.1.

The assumption (a) requires an existence of an ad -invariant subspace $\mathcal{V} \subset \mathcal{A}$, such that the multiplication m in \mathcal{A} induces isomorphisms of vector spaces

$$(a1) \quad m: U_\nu(\mathfrak{g}) \otimes \mathcal{V} \rightarrow \mathcal{A}, \quad (a2) \quad m: \mathcal{V} \otimes U_\nu(\mathfrak{g}) \rightarrow \mathcal{A}.$$

Assumptions (a1) and (a2) are not of equal use. For the Mickelsson algebra $Z^n(\mathcal{A})$ we need the condition (a1) only when we use generators z_v , that is, in Proposition 3.3, Theorem 2, Corollary 4.7, in Section 6.2 and in corresponding statements of Section 9. On the contrary, the construction of the Zhelobenko operators for the algebra $Z^n(\mathcal{A})$ requires the condition (a1) from the very beginning. The condition (a2) is also necessary for the existence of the generators z'_v .

For the algebra $Z_{n-}(\mathcal{A})$ the situation is opposite. We need the condition (a1) for the construction of the Zhelobenko maps and generators \tilde{z}'_v , while the condition (a2) is related only to the generators \tilde{z}_v . Both conditions (a1) and (a2) are satisfied for basic examples, listed in Section 3.1.

The condition (b) requires a local nilpotency of the adjoint action of real root vectors in \mathcal{V} . It always takes place if the space \mathcal{V} is a sum of integrable representations or an affinization $V(z)$ of a locally finite representation of an affine algebra $U'_\nu(\mathfrak{g})$ with the gradation element dropped.

In the latter case the generators z_ν or z'_ν do not formally exist, since neither the highest weight (HW) from Section 3.3 nor the lowest weight (LW) condition from Section 7.1 is satisfied. Nevertheless, in this case the generators of the Mickelsson algebra exist as formal series and could be used with a proper attention to convergences.

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